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المرحلة : الثالثة
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The bonded of probability

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* Markov Inequality

If X is a random variable and if p ($x \geq 0) = 1$ then $p(x \geq t) \leq \frac{E(x)}{t}$,

for $t > 0$

proof :

case "1" : if x is a Continues random variable with a p.d.f f(x)

$since \quad P(X \geq 0) = 1 \rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$

$f(x) > 0$ for $X \geq 0$

=0 O.W

$$E(x) = \int_0^{\infty} xf(x)dx = \int_0^{t} xf(x)dx + \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} xf(x)dx \geq \int_t^{\infty} tf(x)dx$$

∴ E(x)≥ $\int_t^{\infty} tf(x)dx \rightarrow E(x) \geq t \int_t^{\infty} f(x)dx$

$$E(x) \geq t, \qquad p(x \geq t)$$

$\therefore p(x \geq t) \leq \frac{E(x)}{t}$

case "2" : If X is a Discreet random variable with a p.m.f. f(x) by the same method.

** ChebyshV's Inequalities

Let X be random variable where $V(x)$ exists, then :

1- $pr[|X - \mu| \geq t] \leq \frac{V(x)}{t^2}$, $t > 0$

2- $pr[|\bar{x} - \mu| < t] \geq 1 - \frac{V(x)}{t^2}$, $t > 0$

Note:

1- $\frac{V(x)}{t^2}$ is called the upper bound of $p(|x - \mu| \geq t)$.
2- 1- $\frac{V(x)}{t^2}$ is called the lower bound of $p(|x - \mu| \geq t)$.

proof:

1- $V(x) = E[\,(x - \mu)^2 \geq 0$

Let y =$(x-\mu)^2 \geq 0 \rightarrow E(y) = E(x-\mu)^2 = V(x) \geq 0$

by Markov inequality , we get

$p(y \geq t^2) \leq \frac{E(y)}{t^2} \rightarrow p[\,(x-\mu)^2 \geq t^2] \leq \frac{V(x)}{t^2}$

$$p[|x-\mu| \geq t] \leq \frac{V(x)}{t^2}$$

2- $p(A^c) = 1 - p(A)$

$$P[|x-\mu| \geq t^2] = 1 - p[|x-\mu| \ < t^2] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| \ < t^2] \geq 1 - \frac{V(x)}{t^2}$$

Example (1).

If $x \sim unif.(-\sqrt{3}\ ,\sqrt{3}\,)$ then :

a- find the upper bound of $p\left[|x-\mu| \ \geq \frac{3}{2}\right]$

b- find the value of $p\left[|x-\mu| \ \geq \frac{3}{2}\right]$

sol.

a- $f(x) = \begin{cases} \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}} & for -\sqrt{3} < x < \sqrt{3} \\ 0 & o.w \end{cases}$

u.b = $\frac{V(x)}{t^2}$, t = $\frac{3}{2}$

$E(x) = \int_{-\sqrt{3}}^{\sqrt{3}} x\,.\frac{1}{2\sqrt{3}}\ dx = 0$

$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}} dx = \frac{1}{6\sqrt{3}}[\,\sqrt{3}^2 - \left(-\sqrt{3}\right)^2\Big] = 1$

$\therefore V(x) = \ E(x^2) - [E(x)]^2$

$\therefore$ u.b = $\frac{V(x)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9}$

b- $p\left[|x-\mu| \ \geq \frac{3}{2}\right] = p\left(|x-0| \geq \frac{3}{2}\right) =$

$p(|x| \geq \frac{3}{2})$

$= 1- p\left(|x| < \frac{3}{2}\right)$

$=1 - p\left(-\frac{3}{2} < x < \frac{3}{2}\right)$

$= 1- \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$

$= 1- \frac{1}{2\sqrt{3}}\left(\frac{3}{2} + \frac{3}{2}\right)$

$= 1 - \frac{3}{2\sqrt{3}}$

$= 1- \frac{\sqrt{3}}{2} \quad = \ 0.134$

Example (2)

Given a p.d.f. , f(x) = $\begin{cases} \frac{2x}{9} & for\ 0 < x < 3 \\ 0 & o.w. \end{cases}$

a- find the lower bound of p($\frac{5}{4} < x < \frac{11}{4}$)

b- find the value of p($\frac{5}{4} < x < \frac{11}{4}$)

sol .: a- $since \quad L.b. = 1 - \frac{V(x)}{t}$

$$E(x) = \int_0^3 x.\frac{2x}{9}\ dx \ = \frac{2}{27}\ x^3]_0^3$$

$$E(x) = \frac{2}{27}\ (27 - 0) = 2$$

$$E(x^2) = \int_0^3 x^2.\frac{2x}{9}\ dx \ = \frac{2}{36}\ x^4]_0^3$$

$$= \frac{1}{18}\ (81) = \frac{9}{2} = 4.5$$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$$= 4.5 - 4 = 0.5 = \frac{1}{2}$$

$$p\left(\frac{5}{4} < x < \frac{11}{4}\right) = p\left(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2\right)$$

$$= p\left(-\frac{3}{4} < x - 2 < \frac{3}{4}\right)$$

$$= p\left(|x - 2| < \frac{3}{4}\right)$$

$$\therefore \quad t = \frac{3}{4}$$

$$\therefore \quad L.b = 1 - \frac{V(x)}{t} = 1 - \frac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{8}{9} = \frac{1}{9}$$

b- $p\left(\frac{5}{4} < x < \frac{11}{4}\right) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} \, dx = \frac{1}{9} x^2 \Big]_{5/4}^{11/4}$

$$= \frac{1}{9}\left[\frac{121}{16} - \frac{25}{16}\right]$$

$$= \frac{1}{9}\left[\frac{96}{16}\right] = \frac{6}{9}$$