Seventh lecture of Integration Methods

(16) Sixteenth method: involving the square root of quadratic functions method

In this method we can integrate functions from the type $\sqrt{ax^2+bx+c}$ by converting the quadratic function under the root to completing of the square form as in the following examples

Example (58): Evaluate $I=\int \frac{dx}{\sqrt{2x-x^2}}$

$$2x-x^2=-x^2+2x=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1$$

$$=-(x-1)^2+1=1-(x-1)^2$$

$$I=\int \frac{dx}{\sqrt{2x-x^2}}=\int \frac{dx}{\sqrt{1-(x-1)^2}}=\sin^{-1}(x-1)+c$$

Example (59): Evaluate $I=\int \frac{(x-1)dx}{\sqrt{2x-x^2}}$

$$2x-x^2=-x^2+2x=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1$$

$$=-(x-1)^2+1=1-(x-1)^2$$

$$I=\int \frac{(x-1)dx}{\sqrt{2x-x^2}}=\int \frac{(x-1)dx}{\sqrt{1-(x-1)^2}}=\int \left[1-(x-1)^2\right]^{-1/2}(x-1)dx=\int \frac{1}{2}\left[1-(x-1)^2\right]^{-1/2}(2)(x-1)dx$$

$$I=\frac{1}{2}\frac{\left[1-(x-1)^2\right]^{1/2}}{\frac{1}{2}}+c=\left[1-(x-1)^2\right]^{1/2}+c=\sqrt{1-(x-1)^2}+c$$

(17) Seventeenth method: Integration of trigonometric functions using other trigonometric functions

If We can not be integrate trigonometric functions , We can use other trigonometric functions to become the integration is easier than first form by using the following assumptions.

let $z=\tan\frac{x}{2}$

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2z}{1-z^2}$$

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+\tan^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+z^2} - 1 = \frac{2-1-z^2}{1+z^2} = \frac{1-z^2}{1+z^2}$$

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\cos^2\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right)\frac{1}{\sec^2\left(\frac{x}{2}\right)} = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2z}{1+z^2}$$

$$z = \tan\frac{x}{2} \quad \Rightarrow \quad \frac{x}{2} = \tan^{-1} z \quad \Rightarrow \quad x = 2\tan^{-1} z \quad \Rightarrow \quad dx = 2\frac{1}{1-z^2}$$

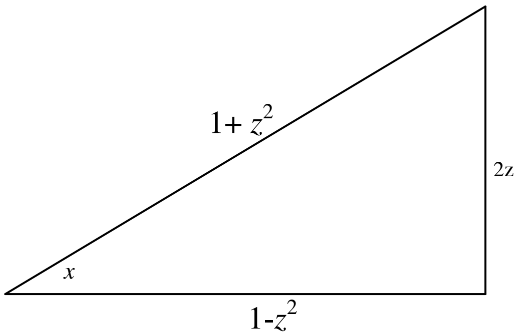
$$dx = \frac{2}{1-z^2}$$

Also we can find the other relations by using the following triangle.

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$dx = \frac{2dz}{1+z^2}$$



Example (60): Evaluate $I = \int \frac{dx}{2+\sin x}$

$$\sin x = \frac{2z}{1+z^2} \quad , \quad dx = \frac{2dz}{1+z^2}$$

$$I = \int \frac{dx}{2+\sin x} = \int \frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{2+2z^2+2z}{1+z^2}} = \int \frac{2dz}{2(z^2+z+1)} = \int \frac{dz}{z^2+z+1}$$

$$z^2+z+1 = z^2+z+\frac{1}{4}+\frac{3}{4} = (z+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2$$

$$I = \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \frac{2}{\sqrt{3}}\tan^{-1}[\frac{z+(1/2)}{(\sqrt{3}/2)}] + c$$

$$I = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{z+(1/2)}{(\sqrt{3}/2)}\right] + c = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\tan(x/2)+(1/2)}{(\sqrt{3}/2)}\right] + c$$

Example (61): Evaluate $I=\int \frac{dx}{1+\cos x}$

$$\cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int \frac{dx}{1+\cos x}=\int \frac{\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}}=\int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2+1-z^2}{1-z^2}}=\int \frac{\frac{2dz}{1+z^2}}{\frac{2}{1-z^2}}=\int dz=z+c=\tan(\frac{x}{2})+c$$

Example (62): Evaluate $I=\int \frac{\sin xdx}{\cos^2 x+\cos x-2}$

$$\sin x=\frac{2z}{1+z^2} \quad , \quad \cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int \frac{\sin xdx}{\cos^2 x+\cos x-2}=\int \frac{\frac{2z}{1+z^2}\cdot\frac{2dz}{1+z^2}}{\left(\frac{1-z^2}{1+z^2}\right)^2+\left(\frac{1-z^2}{1+z^2}\right)-2}$$

$$=\int \frac{\frac{4zdz}{(1+z^2)^2}}{\frac{(1-z^2)^2+(1-z^2)(1+z^2)-2(1-z^2)^2}{(1+z^2)^2}}=\int \frac{4zdz}{1-2z^2+z^4+1-z^4-2-4z^2-2z^2}$$

$$=\int \frac{4zdz}{-6z^2-2z^4}=\int \frac{4zdz}{-2z^2(3+z^2)}=-2\int \frac{1}{z(3+z^2)}dz$$

$$\frac{1}{z(3+z^2)}=\frac{A}{z}+\frac{Bz+C}{3+z^2}=\frac{3A+Az^2+Bz^2+Cz}{z(3+z^2)}=\frac{(A+B)z^2+Cz+3A}{z(3+z^2)}$$

$$\left.\begin{array}{r} A+B=0 \\ C=0 \\ 3A=1 \end{array}\right\} \Rightarrow \begin{array}{c} B=-1/3 \\ C=0 \\ A=1/3 \end{array}$$

$$\frac{1}{z(3+z^2)}=\frac{1/3}{z}+\frac{-1/3z}{3+z^2}$$

$$I=-2\int \frac{1}{z(3+z^2)}dz=-2\int (\frac{1/3}{z}+\frac{-1/3z}{3+z^2})dz=\frac{-2}{3}\ln z+\frac{1}{3}\ln(3+z^2)+c$$

$$I=\frac{-2}{3}\ln(\tan(\frac{x}{2}))+\frac{1}{3}\ln(3+\tan^2(\frac{x}{2}))+c$$

Example (63): Evaluate $I=\int \frac{\cos xdx}{\cos^2 x+\cos x-2}$

$$\cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int\frac{\cos x dx}{\cos^2 x+\cos x-2}=\int\frac{\frac{1-z^2}{1+z^2}\cdot\frac{2dz}{1+z^2}}{\left(\frac{1-z^2}{1+z^2}\right)^2+\left(\frac{1-z^2}{1+z^2}\right)-2}$$

$$=\int\frac{\frac{2(1-z^2)dz}{(1+z^2)^2}}{\frac{(1-z^2)^2+(1-z^2)(1+z^2)-2(1-z^2)^2}{(1+z^2)^2}}=\int\frac{2(1-z^2)dz}{1-2z^2+z^4+1-z^4-2-4z^2-2z^2}$$

$$=\int\frac{2(1-z^2)dz}{-6z^2-2z^4}=\int\frac{2(1-z^2)dz}{-2z^2(3+z^2)}=\int\frac{z^2-1}{z^2(z^2+3)}dz$$

$$\frac{z^2-1}{z^2(z^2+3)}=\frac{A}{z}+\frac{B}{z^2}+\frac{Cz+D}{z^2+3}=\frac{Az(z^2+3)+B(z^2+3)+z^2(Cz+D)}{z^2(z^2+3)}$$

$$=\frac{Az^3+3Az+Bz^2+3B+Cz^3+Dz^2}{z^2(z^2+3)}=\frac{(A+C)z^3+(B+D)z^2+3Az+3B}{z^2(z^2+3)}$$

$$\left.\begin{aligned}A+C&=0\\B+D&=1\\3A&=0\\3B&=-1\end{aligned}\right\}\Rightarrow\begin{aligned}C&=0\\D&=4/3\\A&=0\\B&=-1/3\end{aligned}$$

$$\frac{z^2-1}{z^2(z^2+3)}=\frac{-1/3}{z^2}+\frac{4/3}{z^2+3}$$

$$I=\int\frac{z^2-1}{z^2(z^2+3)}dz=\int\frac{-1/3}{z^2}dz+\int\frac{4/3}{z^2+3}dz=\frac{-1}{3}\int z^{-2}dz+\frac{4}{3}\int\frac{dz}{z^2+3}=\frac{1}{3z}+\frac{4}{3}\int\frac{dz}{z^2+(\sqrt{3})^2}$$

$$=\frac{1}{3z}+\frac{4}{3}\cdot\frac{1}{\sqrt{3}}\tan^{-1}[\frac{z}{\sqrt{3}}]+c=\frac{1}{3\tan(x/2)}+\frac{4}{3\sqrt{3}}\tan^{-1}[\frac{\tan(x/2)}{\sqrt{3}}]+c$$

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Exercise (37) page (311) – book al-samarrai

(1) $\int\frac{\sin x dx}{1+\sin x}$	**(2)** $\int\frac{3dx}{\sin x+\cos x}$	**(3)** $\int\frac{dx}{1+\sin x-\cos x}$
(4) $\int\frac{\sin x\cos x}{1-\cos x}dx$	**(5)** $\int\frac{dx}{5+4\sin x}$	**(6)** $\int\frac{dx}{3-2\cos x}$
(7) $\int\frac{dx}{\sqrt{x}-\sqrt[4]{x}}$	**(8)** $\int\frac{dx}{x\sqrt{1-x}}$	**(9)** $\int x^5\sqrt{1-x^2}dx$
(10) $\int\frac{\sqrt{x}dx}{1+x}$	**(11)** $\int\frac{dx}{3+\sqrt{x+2}}$	**(12)** $\int_{\pi/2}^{\pi}\frac{dx}{1-\cos x}$

(13) $\int_{0}^{\pi/2} \frac{dx}{2+\cos x}$	**(14)** $\int_{0}^{3} \frac{xdx}{\sqrt{1+x}}$	**(15)** $\int_{1}^{64} \frac{dx}{\sqrt[3]{x}+2\sqrt{x}}$

Exercise (6-12) page (414) blue book

(1) $\int \frac{dx}{1+\cos x}$	**(2)** $\int \frac{dx}{13+5\cos x}$	**(3)** $\int \frac{dx}{\sin x-\cos x}$
(4) $\int \sqrt{1-\sin x}dx$	**(5)** $\int \frac{\cot xdx}{1-\cos x}$	**(6)** $\int \frac{\cos xdx}{2-\cos x}$
(7) $\int \frac{dx}{5+4\sin x}$	**(8)** $\int \frac{dx}{\sin x+\tan x}$	**(9)** $\int \frac{dx}{1+\sin x+\cos x}$

Calculus-12 exercise (8.5) - page 467.