جامعة تكريت

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قسم الرياضيات

المرحلة الثانية

مادة التفاضل المتقدم

المتسلسلات غير المنتهية Infinite Series

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Theorem: The following six sequences converge to the limits listed below:

1- $\lim_{n\to\infty} \frac{lnn}{n} = 0.$
2- $\lim_{n\to\infty} \sqrt[n]{n} = 1.$
3- $\lim_{n\to\infty} x^{\frac{1}{n}} = 1\ (x > 0).$
4- $\lim_{n\to\infty} x^n = 0\ (\ |x| < 1).$
5- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$(for any x).
6- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$(for any x).

Proof: 1), 2), 3)it's easy.

4) If $|x| < 1, \lim_{n\to\infty} x^n = 0$, we need to show that to each $\epsilon > 0$, there corresponds an integers N so large that $|x^n| < \epsilon\ \forall n > N$.

Since $\epsilon^{\frac{1}{n}} \to 1$, while $|x| < 1$, there exists an integer N for which $\epsilon^{\frac{1}{N}} > |x|$,

$|x^N| = |x|^N < \epsilon$ … (1)

If $|x| < 1$, then $|x^n| < |x^N|\ \ \forall n > N$ … (2).

Combing (1) & (2) produces $|x^n| < \epsilon\ \ \forall n > N$.

5) $\forall\ x$, $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. Let $a_n = \left(1 + \frac{x}{n}\right)^n$, then $lna_n = nln\left(1 + \frac{x}{n}\right)$, by L'Hopital's Rule

$\lim_{n\to\infty} nln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{x}{n}}\right)\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$, by theorem with $f(x) = e^x$, we get $a_n = e^{lna_n} = \left(1 + \frac{x}{n}\right)^n \to e^x$.

6) $\forall\ x$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, since $-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$. if $\frac{|x|^n}{n!} \to 0$, by sandwich theorem, we can prove that . Let $M > |x|\ \Rightarrow \frac{|x|}{M} < 1$, by (4) we have that $\left(\frac{|x|}{M}\right)^n \to 0$ s.t $n > M$.

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1\cdot 2\cdot 3 \cdots M\cdot(M+1)\cdots n} \le \frac{|x|^n}{M!\,M^{n-M}} = \frac{|x|^n M^M}{M!\,M^n} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$$

Thus $0 \le \frac{|x|^n}{n!} \le \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$. $\therefore$ $\frac{|x|^n}{n!} \to 0$, becauce $\left(\frac{|x|}{M}\right)^n \to 0$.
By sandwich theorem, we have that proof.

Ex: Test the Conv. Or Div. of the following Sequences.

1- $a_n = \frac{lnn^2}{n} \to \lim\limits_{n\to\infty} \frac{2lnn}{n} = 0$. Conv.

2- $a_n = \left(\frac{n-2}{n}\right)^n \to \lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n = e^{-2}$. Conv.

3- $a_n = n^2 \to \lim\limits_{n\to\infty} n^2 = \infty$. Div.

4- $a_n = \frac{10^n}{n!} \to \lim_{n\to\infty}\frac{10^n}{n!} = 0$. Conv.

Bounded Monotonic Sequences

Def: A sequence $\{a_n\}$ is bounded from above if $\exists\, M \in R$ s.t $a_n \le M\ \forall n$, The number M is an upper bonded for $\{a_n\}$. If M is upper bounded for $\{a_n\}$, but no number less than M is an upper bounded for $\{a_n\}$, then M is the least upper bounded for $\{a_n\}$.

We say $\{a_n\}$ is bounded from below if $\exists\, m \in R$ s.t $a_n \ge m\ \forall n$, The number m is lower bonded for $\{a_n\}$. If m is alower bounded for $\{a_n\}$, but no number greater than m is an lower bounded for $\{a_n\}$, then m is the greatest lower bounded for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is bounded. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an unbounded Sequence.

Def: A sequence $\{a_n\}$ is non-decreasing. If $a_n \le a_{n+1}\ \forall n$. That is $a_1 \le a_2 \le a_3 \le \cdots$, the sequence is non-increasing. If $a_n \ge a_{n+1}\ \forall n$. The sequence $\{a_n\}$ is monotonic if it's either non-decreasing or non-increasing.

Theorem: If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence is converges.

Ex: Test the increasing and decreasing of the following sequences:

1) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ 2) $\left\{\frac{e^n}{(n+1)!}\right\}_{n=1}^{\infty}$ 3) $\{\tan^{-1}(n)\}_{n=1}^{\infty}$.

Sol: 1) $a_{n+1} = \frac{n+1}{n+2}$, $\frac{a_{n+1}}{a_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$. ∴ The sequence is increasing for all n $(n \geq 1)$.

2) $a_{n+1} = \frac{e^{n+1}}{(n+2)!}$, $\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+2)(n+1)!} \cdot \frac{(n+1)!}{e^n} = \frac{e}{n+2} < 1$. ∴ The sequence is decreasing for all n $(n \geq 1)$.

3) $f(n) = a_n = \tan^{-1}(n)$. $f'(n) = \frac{1}{n^2+1} > 0$increasing for $\forall\, n \geq 1$.

Infinite Series

Def: Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an infinite series. The number a_n is the n^th term of the series. The sequence $\{S_n\}$ defined by

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

⋮

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Is the sequence of partial sums of the series, the number S_n being the n^th partial sum. If the sequence of partial sums converges to a limit L, we say that the series converges and that is sum is L. In this case, we also write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Ex: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$

& $1 + 2 + 3 + 4 + 5 + \cdots = \sum_{n=1}^{\infty} n = \infty.$

Geometric Series

The geometric series is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n \ , a \neq 0$$

The sum of the first n[th] terms of formula above is

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \ldots$ (1)

Multiply both sides of Eq.(1) by r, we have that

$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n \ \ldots$ (2)

Subtract Eq.(2) from (1). Thus $S_n - rS_n = a - ar^n$.

$\therefore \ S_n = \frac{a(1-r^n)}{1-r}$, s.t $r \neq 1$.

Now to test the convergence of this series

$$\lim_{n\to\infty} S_n = S_0 \ \Rightarrow \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{a}{1-r} - \lim_{n\to\infty} \frac{ar^n}{1-r}$$

- $S_n = \begin{cases} na & r = 1 \\ \mp a & r = -1 \\ \infty & r > 1 \end{cases} \to \infty$, as $n \to \infty$. Div.
- If $|r| < 1$, as $n \to \infty$. $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$.

$$\therefore \ S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad , |r| < 1$$

Ex: Test the Conv. and Div. of the following series and find of the sum:

1) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ 2) $\sum_{n=0}^{\infty} 3^n$ 3) $\sum_{n=0}^{\infty} \frac{3^n+2^n}{5^n}$.

Sol: 1) $= 1$, $r = \frac{1}{2} < 1$, $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$.

2) $a = 1$, $r = 3 > 1$, Div.

3) $\sum_{n=0}^{\infty} \frac{3^n+2^n}{5^n} = \sum_{n=0}^{\infty} \frac{3^n}{5^n} + \frac{2^n}{5^n}$, since $\frac{3}{5} < 1$ & $\frac{2}{5} < 1$ with $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$,

$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$,then $\sum_{n=0}^{\infty} \frac{3^n+2^n}{5^n} = \frac{5}{2} + \frac{5}{3} = \frac{25}{6}$.

Ex: Find the sum of the following series

1) $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots,\ a = 5, r = \left|-\frac{1}{4}\right| < 1.$

$\therefore S = \frac{5}{1+\frac{1}{4}} = 4.$

2) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$, since $\left|\frac{1}{2}\right| < 1, \left|\frac{1}{6}\right| < 1$, then $A = \frac{1}{1-\frac{1}{2}} = 2$,

$B = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$

Then $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right) = A - B = 2 - \frac{6}{5} = \frac{4}{5}$.

3) $\sum_{n=0}^{\infty} \frac{4}{2^n},\ a = 4, r = \left|\frac{1}{2}\right| < 1, \therefore S = \frac{4}{1-\frac{1}{2}} = 8.$

4) $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - \cdots$, $\{S_n\} = \{1,0,1,0,\ldots\}$ is Div. because of the oscillation of S_n between 1,0.

Theorem: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series then:

1) Sum Rule: $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B.$
2) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA.$