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كلية التربية للبنات

قسم الرياضيات

المرحلة الثانية

مادة التفاضل المتقدم

# المتتابعات غير المنتهية Infinite sequences

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**Definition:** A sequence is a function from the positive integers numbers to real numbers. and we denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$,

A sequence is a list of numbers written in a definite order $\{a_1, a_2, a_3, \dots, a_n, \dots\}$.

The Sequence have, $a_1$ is called the first term, $a_2$ the second term, $a_n$ is the n[th] term.

can be dfind by given a formula for n[th] term as shown that:-

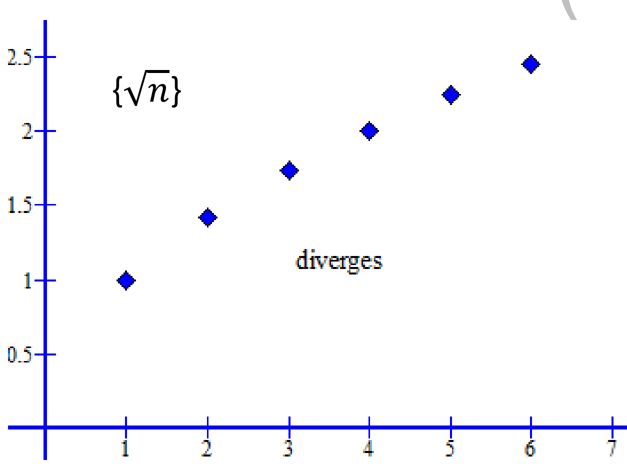
1- $\{\frac{n}{n+1}\}_{n=1}^{\infty} \to \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\}$.

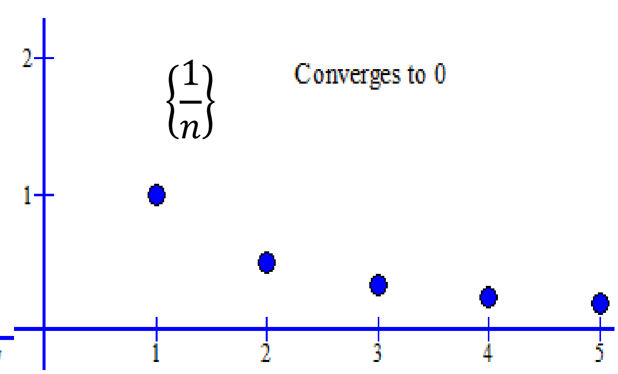
2- $\{\frac{(-1)^n(n+1)}{3^n}\}_{n=1}^{\infty} \to \{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{8}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\}$.

3- $\{Cos(\frac{n\pi}{6})\}_{n=1}^{\infty} \to \{1, \frac{\sqrt{3}}{2}, 0, \dots, Cos\left(\frac{n\pi}{6}\right), \dots\}$.

4- $\{\sqrt{n-3}\}_{n=1}^{\infty} \to \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$.

Convergence and divergence (Graphically):-



**Definition:** The sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number $\mathcal{L}$, if for every positive number $\epsilon$ there corresponds an integer N such that for all n.

$n > N \; \rightarrow \; |a_n - \mathcal{L}| < \epsilon.$

If no such number $\mathcal{L}$ exists, we say that $\{a_n\}$ diverges. If $\{a_n\}$ converges to $\mathcal{L}$, we write $\lim_{n\to\infty} a_n = \mathcal{L}$ or simply $a_n \rightarrow \mathcal{L}$, and called $\mathcal{L}$ the limit of the sequence.

**Ex:** Show that: 1- $\lim_{n\to\infty} \frac{1}{n} = 0$ 2- $\lim_{n\to\infty} k = k.$

Sol: 1) Let $\epsilon > 0$ be given, we must show that $\exists N \in Z$ s.t $n > N \rightarrow \left|\frac{1}{n} - 0\right| < \epsilon \Rightarrow \frac{1}{n} < \epsilon$or $n > \frac{1}{\epsilon}$. If N is any integer greater than $\frac{1}{\epsilon}$, will hold $\forall\, n > N$. Then $\lim_{n\to\infty} \frac{1}{n} = 0.$

2) Let $\epsilon > 0$ be given, $n > N \rightarrow |k - k| < \epsilon$, since $k - k = 0$, this all ready it's hold $\forall\, n > N$. Then $\lim_{n\to\infty} k = k.$

**Ex:** Solve of the following sequences.

**1-** $\{\frac{n^2}{(n+1)^2}\}$ 2- $\{\frac{n^2}{2n-1} Sin(\frac{1}{n})\}$ 3- $\{\frac{lnn}{n}\}$.

Sol:

1) $\lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$, the sequence convergence to 1.

2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n-1} Sin(\frac{1}{n})$, let u=1/n, where $n \rightarrow \infty$, $u \rightarrow 0$

$\lim_{u\to 0} \frac{\frac{1}{u^2}}{2\frac{1}{u}-1} \sin(u) = \lim_{u\to 0} \frac{1}{2-u} \cdot \frac{\sin(u)}{u} = \frac{1}{2}$. The sequence converges to $\frac{1}{2}$.

3) $\lim_{n\to\infty} \frac{lnn}{n}$, by using L'Hopital's Rule $=\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$. The Seq. Conv. To 0.

**Theorem:** Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1- Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$.

2- Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$.

3- Constant Multiple Rule: $\lim_{n\to\infty}(kb_n) = kB, k \in R$.

4- Product Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$.

5- Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ , if $B \neq 0$.

**Theorem:** (The Sandwich theorem for Sequences)

Let $\{a_n\}$ ,$\{b_n\}$ and $\{C_n\}$ be sequences of real numbers. If $a_n \le b_n \le C_n$ hold for all n beyond some index N, and if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} C_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

**Ex:** Since $\frac{1}{n} \to 0$, we know that

1- $\frac{Cos(n)}{n} \to 0$ , because $-\frac{1}{n} \le \frac{Cos(n)}{n} \le \frac{1}{n}$.

2- $\frac{1}{2^n} \to 0$ , because $0 \le \frac{1}{2^n} \le \frac{1}{n}$.

3- $(-1)^n \frac{1}{n} \to 0$ , because $-\frac{1}{n} \le (-1)^n \frac{1}{n} \le \frac{1}{n}$.

**Theorem:** Let $\{a_n\}$ be sequence of real numbers. If $a_n \to L$ and if **f** is a function; that is continuous at $L$ and defined at all $a_n$, then $f(a_n) \to f(L)$.

**Ex:** Show that $\sqrt{\frac{n+1}{n}} \to 1$.

Sol: We know that $\frac{n+1}{n} \to 1$, taking $f(x) = \sqrt{x}$ & L=1 in theorem gives $\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$

- $\{2^{\frac{1}{n}}\} \to 1, f(x) = 2^x$ , $a_n = \frac{1}{n}$.

**Theorem:** Suppose that $f(x)$ is a function defined $\forall\, x \geq n_0$ , and that $\{a_n\}$ is a sequence of real numbers s.t $a_n = f(n), \forall\, n \geq n_0$ .

Then $\lim_{x\to\infty} f(x) = L \;\; \Rightarrow \;\; \lim_{n\to\infty} a_n = L.$

**Ex:** Show that $\lim_{n\to\infty} \frac{lnn}{n} = 0$

The function $\frac{lnx}{x}$ is defined $\forall\, x \geq 1 \Rightarrow \lim_{x\to\infty} \frac{lnx}{x}$

by L'Hopital'sRule $\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0.$

$\therefore \lim_{n\to\infty} \frac{lnn}{n} = 0.$

**Ex:** Does the Sequence whose n[th] term is converge? If so, find $\lim_{n\to\infty} a_n$.

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

Sol: $ln\, a_n = nln\left(\frac{n+1}{n-1}\right)$, then $\lim_{n\to\infty} ln\, a_n = \lim_{n\to\infty} nln\left(\frac{n+1}{n-1}\right)$ , by L'Hopital's Rule

$\lim_{n\to\infty} \frac{ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{-2}{(n-1)^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{(n-1)^2}$=2.

Since $a_n \to 2$, and $f(x) = e^x$ is Cont.

$\therefore\; a_n = e^{ln a_n} \;\to e^2.$