Interpolation and Extrapolation

Lecture Notes

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Chapter 1

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1.1 Introduction

In applied sciences and engineering, scientists and engineers often collect a number of data points of different scientific phenomena via experimentation and sampling. In many cases they need to estimate (interpolate) a function at a point its functional value is not in the range of the collected data. Interpolation is a branch of numerical analysis studies the methods and techniques of estimating an unknown value of a function at an intermediate value of its independent variable. Also, interpolation is used to replace a complicated function by a simpler one.

1.2 Lagrange Interpolation

Suppose that we would like to interpolate an arbitrary function f at a set of limited points $x_0, x_1, \cdots, x_n$. These $n+1$ points are known as **interpolation nodes** in interpolation theory. Firstly, we need to introduce a system of $n+1$ special polynomials of degree n known as **interpolating polynomials** or **cardinal polynomials**. These polynomials are denoted by $\ell_0, \ell_1, \cdots, \ell_n$ and defined using Kronecker delta notation as follows

$$\ell_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

.

Then, we can interpolate the function f by a polynomial P_n of degree n defined by

$$P_n(x) = \sum_{i=0}^{n} \ell_i(x) f(x_i),$$

this polynomial is called **Lagrange polynomial** or **Lagrange form of the interpolation polynomial**, and it is a linear combination of the cardinal polynomials $\ell_i, i = 0, 1, \cdots, n$. Moreover, it coincides with the function f at the the nodes $x_j, j = 0, 1, \cdots, n$, namely

$$P_n(x_j) = \sum_{i=0}^{n} \ell_i(x_j) f(x_j) = \ell_j(x_j) f(x_j) = f(x_j).$$

The interpolating polynomials can be expressed as a product of n linear factors

$$\ell_i(x) = \prod_{j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)},\ i = 0, 1, \cdots, .$$

i.e.

$$\ell_i(x) = \frac{(x - x_0)}{(x_i - x_0)} \frac{(x - x_1)}{(x_i - x_1)} \cdots \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \frac{(x - x_{i+1})}{(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}.$$

Example 1. *Determine the linear Lagrange polynomial that passes through the points* $(1, 5)$ *and* $(4, 2)$ *and use it to interpolate the linear function at* $x = 3$.

Solution: Writing out the cardinal polynomials

$$\ell_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} = \frac{(x - 4)}{(1 - 4)} = \frac{-1}{3}(x - 4),$$

and

$$\ell_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} = \frac{(x - 1)}{(4 - 1)} = \frac{1}{3}(x - 1).$$

Hence, the Lagrange polynomial is

$$P_1(x) = \sum_{i=0}^{1} \ell_i(x) f(x_i) = \ell_0(x) f(x_0) + \ell_1(x) f(x_1) =$$

$$\frac{-1}{3}(x - 4)(5) + \frac{1}{3}(x - 1)(2) = -x + 6.$$

So,

$$P_1(3) = -(3) + 6 = 3.$$

Note that

$$P_1(1) = -(1) + 6 = 5 = f(1), \text{ and } P_1(4) = -(4) + 6 = 2 = f(4).$$

Example 2. *Find the Lagrange polynomial that interpolates the following data*

x	1	2	2.5	3	4	5
$f(x)$	0	5	6.5	7	3	1

Solution: The cardinal polynomials are:

$$\begin{aligned}
\ell_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} \\
&= \frac{(x-2)(x-2.5)(x-3)(x-4)(x-5)}{(1-2)(1-2.5)(1-3)(1-4)(1-5)} \\
&= -\frac{1}{36}x^5 + \frac{11}{24}x^4 - \frac{53}{18}x^3 + \frac{221}{24}x^2 - \frac{505}{36}x + \frac{25}{3},
\end{aligned}$$

$$\begin{aligned}
\ell_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} \\
&= \frac{(x-1)(x-2.5)(x-3)(x-4)(x-5)}{(2-1)(2-2.5)(2-3)(2-4)(2-5)} \\
&= \frac{1}{3}x^5 - \frac{31}{6}x^4 + \frac{61}{2}x^3 - \frac{509}{6}x^2 + \frac{655}{6}x - 50,
\end{aligned}$$

$$\begin{aligned}
\ell_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} \\
&= \frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(2.5-1)(2.5-2)(2.5-3)(2.5-4)(2.5-5)} \\
&= -\frac{5000}{7031}x^5 + \frac{75000}{7031}x^4 - \frac{425000}{7031}x^3 + \frac{1125000}{7031}x^2 - \frac{1370000}{7031}x + \frac{600000}{7031},
\end{aligned}$$

$$\begin{aligned}
\ell_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} \\
&= \frac{(x-1)(x-2)(x-2.5)(x-4)(x-5)}{(3-1)(3-2)(3-2.5)(3-4)(3-5)} \\
&= \frac{1}{2}x^5 - \frac{29}{4}x^4 + \frac{79}{2}x^3 - \frac{401}{4}x^2 + \frac{235}{2}x - 50,
\end{aligned}$$

$$\begin{aligned}
\ell_4(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} \\
&= \frac{(x-1)(x-2)(x-2.5)(x-3)(x-5)}{(4-1)(4-2)(4-2.5)(4-3)(4-5)} \\
&= -\frac{1}{9}x^5 + \frac{3}{2}x^4 - \frac{137}{18}x^3 + \frac{109}{6}x^2 - \frac{365}{18}x + \frac{25}{3},
\end{aligned}$$

$$\begin{aligned}
\ell_5(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} \\
&= \frac{(x-1)(x-2)(x-2.5)(x-3)(x-4)}{(5-1)(5-2)(5-2.5)(5-3)(5-4)} \\
&= \frac{1}{60}x^5 - \frac{5}{24}x^4 + x^3 - \frac{55}{24}x^2 + \frac{149}{60}x - 1.
\end{aligned}$$

Hence, the Lagrange polynomial is

$$P_5(x) = \sum_{i=0}^{6} \ell_i(x)f(x_i) = \ell_i(x)f(x_i) + \ell_i(x)f(x_i) +$$
$$\ell_i(x)f(x_i) + \ell_i(x)f(x_i) + \ell_i(x)f(x_i) + \ell_i(x)f(x_i) =$$
$$-\frac{1}{36}x^5 + \frac{11}{24}x^4 - \frac{53}{18}x^3 + \frac{221}{24}x^2 - \frac{505}{36}x + \frac{25}{3}\,(0) +$$
$$\frac{1}{3}x^5 - \frac{31}{6}x^4 + \frac{61}{2}x^3 - \frac{509}{6}x^2 + \frac{655}{6}x - 50\,(5) +$$
$$-\frac{5000}{7031}x^5 + \frac{75000}{7031}x^4 - \frac{425000}{7031}x^3 + \frac{1125000}{7031}x^2 - \frac{1370000}{7031}x + \frac{600000}{7031}\,(6.5) +$$
$$\frac{1}{2}x^5 - \frac{29}{4}x^4 + \frac{79}{2}x^3 - \frac{401}{4}x^2 + \frac{235}{2}x - 50\,(7) +$$
$$-\frac{1}{9}x^5 + \frac{3}{2}x^4 - \frac{137}{18}x^3 + \frac{109}{6}x^2 - \frac{365}{18}x + \frac{25}{3}\,(3) +$$
$$\frac{1}{60}x^5 - \frac{5}{24}x^4 + x^3 - \frac{55}{24}x^2 + \frac{149}{60}x - 1\,(1).$$

After some mathematical manipulation, we have

$$P_5(x) = -\frac{8}{316395}x^5 + \frac{8}{21093}x^4 - \frac{2722272566677}{1266637395197952}x^3 + \frac{200167100491}{35184372088832}x^2$$
$$-\frac{10969157106929}{1583296743997440}x - \frac{187476506320011}{8796093022208}.$$

Note that the Lagrange interpolant is used to interpolate a function at a set of non-equally spaced points.

1.3 Newton's Difference Interpolation Formula

Newton's interpolation formula is used to interpolate a function at a set of given equally spaced points $x_0, x_1, \cdots, x_n$. Before we start we need to define the finite divided differences.

1.3.1 Finite Divided Differences

The first finite divided difference of the function f is in general given by

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}.$$

The second finite divided difference is the difference between the two divided difference, is represented by

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i}.$$

Likewise, the nth finite divided difference is expressed by

$$f[x_0, x_1, \cdots, x_{n-1}, x_n] = \frac{f[x_1, \cdots, x_{n-1}, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}.$$

Note that the zero-order difference is defined as

$$f[x_i] = f(x_i) = f_i.$$

Also, observe that

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i} = \frac{f(x_i) - f(x_j)}{x_i - x_j} = f[x_j, x_i].$$

The divided differences is summarised in the divided difference table given below:

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	f_0	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	f_1	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	$f[x_1, x_2, x_3, x_4]$
x_2	f_2	$f[x_2, x_3]$	$f[x_2, x_3, x_4]$	
x_3	f_3	$f[x_3, x_4]$		
x_4	f_4			

Example 3. *Compute the divided differences of the following data*

x	0.5000	1.000	1.500	2.000	2.5000
$f(x)$	1.1250	3.000	7.3750	15.0000	26.6250

Solution: Using the standard notation the first finite divided differences are:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{3 - 1.1250}{1 - 0.5} = \frac{1.8750}{0.5} = 3.7500.$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{7.3750 - 3}{1.5 - 1} = \frac{4.3750}{0.5} = 8.7500.$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{15 - 7.3750}{2 - 1.5} = \frac{7.6250}{0.5} = 15.2500.$$

$$f[x_3, x_4] = \frac{f(x_4) - f(x_3)}{x_4 - x_3} = \frac{26.6250 - 15}{2.5 - 2} = \frac{11.6250}{0.5} = 23.2500.$$

Now, using the computed first divided differences, we compute the second divided differences

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{8.7500 - 3.7500}{1.5 - 0.5} = \frac{5}{1} = 5.000.$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{8.7500 - 3.7500}{1.5 - 0.5} = \frac{5}{1} = 5.000.$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{15.2500 - 8.7500}{2 - 1} = \frac{6.5000}{1} = 6.5000.$$

$$f[x_2, x_3, x_4] = \frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = \frac{23.2500 - 15.2500}{2.5 - 1.5} = \frac{8.0000}{1} = 8.0000.$$

Finally, we compute the third divided differences using the computed second divided differences

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{6.5000 - 5.0000}{2 - 0.5} = \frac{1.5000}{1.5000} = 1.0000.$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = \frac{8.0000 - 6.5000}{2.5 - 1} = \frac{1.5000}{1.5000} = 1.0000.$$

The results are outlined in the following table

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0.5000	1.1250	3.7500	5.000	1.0000
1.000	3.000	8.7500	6.5000	1.0000
1.5000	7.3750	15.2500	8.0000	
2.000	15.0000	23.2500		
2.5000	26.6250			

1.3.2 Newton's Interpolation Divided Difference Formula

The general form of Newton's interpolation polynomial of order n for $n+1$ data points is

$$P_n(x) = d_0 + d_1(x-x_0) + d_2(x-x_0)(x-x_1) + d_3(x-x_0)(x-x_1)(x-x_2) + \cdots + d_n(x-x_0)(x-x_1) + d_3(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}),$$

where

$$d_0 = f[x_0],$$

$$d_1 = f[x_0, x_1],$$

$$d_2 = f[x_0, x_1, x_2],$$

$$d_3 = f[x_0, x_1, x_2, x_3],$$

$$\vdots$$

$$d_n = f[x_0, x_1, \cdots, x_n],$$

Example 4. *Use the data from Example 3 to construct Newton's interpolation divided difference formula, and use it to evaluate $f(0)$, $f(3)$, and $f(3.25)$.*

Solution: The Newton's polynomial of third order for the data in the table above is

$$P_3(x) = d_0 + d_1(x - x_0) + d_2(x - x_0)(x - x_1) + d_3(x - x_0)(x - x_1)(x - x_2) = 1.1250 + 3.75(x - 0.5) + 5(x - 0.5)(x - 1) + (x - 0.5)(x - 1)(x - 1.5).$$

After some mathematical manipulation, we have

$$P_3(x) = x^3 + 2x^2 - x + 1.$$

Hence,

$$f(0) = P_3(0) = 1,\ f(3) = P_3(3) = 43,\ f(3.25) = P_3(3.25) = 53.2031.$$

1.4 Extrapolation

In numerical analysis, **extrapolation** is used to estimate a value of a function at a point beyond the range of its known values $x_0, x_1, \cdots, x_n$. Extrapolation compared to interpolation is more likely to produce meaningless results. There are many methods in extrapolation, in these notes, we consider the linear and polynomial extrapolation.

1.4.1 Linear Extrapolation

Linear extrapolation is used to estimate an approximately linear function by extending its graph not far away from its known values. Assume that we have a set of values of some unknown function f at some points in its domain. Let the function f has values y_{n-1} and y_n at the points x_{n-1} and x_n respectively. We can estimate the function value y at the point x near the points x_{n-1} and x_n by constructing a tangent line to the data points (x_{n-1}, y_{n-1}) and (x_n, y_n) to obtain

$$y(x) = y_{n-1} + \frac{x - x_{n-1}}{x - x_n}(y_n - y_{n-1}).$$

Note that when $x_{n-1} < x < x_n$ then the extrapolation is turned to interpolation process.

Example 5. *Use the following two points* $(0.1, 1.1)$ *and* $(0.35, 1.35)$ *to extrapolate the value of the unknown function* f *at* $x = 0$.

Solution: The general formula of the linear extrapolation is

$$y(x) = y_1 + \frac{x - x_1}{x - x_2}(y_2 - y_1).$$

Here $x_1 = 0.1, y_1 = 1.1, x_2 = 0.35$, and $y_2 = 1.35$.

Substituting these values in the linear extrapolation formula, we have

$$y(0) = 1.1 + \frac{0 - 0.1}{0 - 0.35}(1.35 - 1.1) = 1.1 + \frac{0.1}{0.35}(0.25) = 1.1714.$$

1.4.2 Polynomial Extrapolation

To approximate a function by a high order polynomial near the end of a given set of data or at a point beyond the original observed values, Lagrange interpolation or Newton interpolation can be used to extrapolate the resulting polynomial at the required data. Care has to be taken since the extrapolation error will grow due to the **Runge's Phenomenon**.

Example 6. *Use the data in the table below to extrapolate* $f(600)$

x	300	400	500
$f(x)$	0.616	0.525	0.457

Solution: Constructing the divided difference table of the data

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
300	0.616	-0.00091	0.00000115
400	0.525	-0.00068	
500	0.457		

From the table in above, we can write out the Newton's difference polynomial as follows:

$$P_2(x) = d_0 + d_1(x - x_0) + d_2(x - x_0)(x - x_1) =$$
$$0.616 - 0.00091(x - 300) + 0.00000115(x - 300)(x - 400).$$

After some simplifications, we have

$$P_2(x) = 0.00000115\,x^2 - 0.00175\,x + 1.0270.$$

So

$$f(300) \approx P_2(300) = 0.00000115(300)^2 - 0.00175(300) + 1.0270 = 0.391.$$