Open Methods for Solving Nonlinear Equations

Lecture Notes

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Chapter 1

Open Methods for Solving Nonlinear Equations

In these methods we do not need to have an interval around the root to start solving the nonlinear equation $f(x) = 0$. We only need a sufficiently closed approximation to the root hence, the name **open methods** or **locally convergent methods**. The starting approximation is called the **starting value**, **initial approximation**, **initial guess** or a **seed**. The bracketing methods are slow compared with open methods which are faster and have better convergence properties. In this section we study the fixed-point method, Newton's method and the secant method as examples of open methods. In the literature, the open methods are also known as **slope methods** since these methods use the slope of the tangent line of the graph of the function near the point $(r, 0)$ to derive a mathematical formula for computing the next iterations.

1.0.1 Fixed Point Method

It is an important and widely used method for finding the roots of nonlinear problems. This method relies on the iteration principle. Iteration is a fundamental concept in computer sciences and numerical analysis and is used for solving a wide variety of problems. Iteration and fixed point methods have many applications in *fractals (fractal geometry)*, *chaos theory* and *dynamical systems.*There is a strong connection between *root-finding problems* and *fixed point problems*, and in this section, we use fixed point problems to solve root-finding problems.

Definition 1 (Fixed Point)**.** *The number r is called a **fixed point** of the function g if $r = g(r)$.*

We start by transforming the root-finding problem $f(x) = 0$ to a fixed point problem $x = g(x)$ by algebraic manipulations. There are more than one way of rearranging $f(x) = 0$ into an equivalent form $x = g(x)$. Note that if r is a zero of the function f (i.e. r is a root of the equation $f(r) = 0$) then $r = g(r)$ i.e. r is a fixed point of the function g. Conversely, if g has a fixed point at r then the function $f(x) = x - g(x)$ has a zero at r. Geometrically, the fixed points of a function $y = g(x)$ are the points of intersection of its curve with the straight line $y = x$.

Example 1. *Find the fixed points of the function $g(x) = 2 - x^2$ and verify that they are the solutions to the equation $f(x) = x - g(x) = 0$.*

Solution : The fixed points of g are the points satisfying the fixed point equation $x = g(x)$, so intersect the graph of $y = g(x)$ with the graph of the straight line $y = x$

$$x = g(x) = 2 - x^2,$$

which implies that

$$-x^2 - x + 2 = -(x^2 + x - 2) = -(x - 1)(x + 2) = 0.$$

So, either $(x - 1) = 0$ implies $x = 1$ or $(x + 2) = 0$ implies $x = -2$. Hence, the fixed points are $x = 1$ and $x = -2$. We notice that these fixed points are the same the zeros of $f(x) = x - g(x) = -(x^2 + x - 2) = -(x - 1)(x + 2) = 0$.

Definition 2 (**Fixed Point Iteration**). *The iteration $r_{n+1} = g(r_n)$, $n = 0, 1, \cdots$, obtained by using fixed point formula $x = g(x)$ is called a* ***fixed point iteration*** *or* ***functional iteration****. The numbers $r_n, n = 0, 1, \cdots$, are called* ***iterates*** *or* ***iterations***

In short, in the fixed point method we start with starting value r_0 and by using the repeated substitutions in the rule or function $g(x)$ we compute the successive or consecutive terms. For this reason the fixed-point method sometimes is referred to as **repeated substitution method**.

Theorem 1 (**Convergence of the Fixed Point Iteration**). *Let g is a continuous function and $\{r_n\}_{n=0}^{\infty}$ is a sequence of iterates generated by the fixed-point iteration rule $r_{n+1} = g(r_n), n = 0, 1, \cdots$. If the sequence $\{r_n\}_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} r_n = r$, then r is a fixed point of the function $g(x)$.*

Theorem 2 (**Existence and Uniqueness of the Fixed Point**). *Assume that $g \in C[a, b]$.*

1. *If $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a at least one fixed point r in $[a, b]$.*

2. *If also, $g'(x)$ existed and defined on (a,b) and there exists a positive constant $K < 1$ such that $|g'(x)| \leq K < 1$, for all $x \in (a,b)$, then g has a unique fixed point r in $[a,b]$.*

3. *If g satisfies the conditions (1) and (2) , then for any number r_0 in $[a,b]$ the sequence $\{r_n\}_{n=0}^{\infty}$ of iterations generated by fixed point iteration $r_{n+1} = g(r_n)$, $n = 0, 1, \cdots$, converges to the unique fixed point r in $[a,b]$.*

Theorem 3 (**Fixed Point Theorem**). *Assume that*

(i) $g, g' \in C[a,b]$.

(ii) K *is a positive constant.*

(iii) $r_0 \in (a,b)$ *is an initial approximation.*

(iv) $g(x) \in [a,b]$ *for all* $x \in [a,b]$.

1. *If $|g'(x)| \leq K < 1$ for all $x \in (a,b)$, then the sequence of iterates $\{r_n\}_{n=0}^{\infty}$ converges to the unique fixed point $r \in [a,b]$ and r is called an **attractive fixed point**.*

2. *If $|g'(x)| > 1$ for all $x \in (a,b)$, then the sequence of iterates $\{r_n\}_{n=0}^{\infty}$ diverges and will not converge to the fixed point $r \in [a,b]$ and r is called a **repelling fixed point** and the iteration exhibits local divergence.*

Corollary 1 (**Fixed Point Iteration Error Bounds**). *If g satisfies the hypotheses of Fixed Point Theorem, then the error bounds for approximating r using r_n are given by*

$$|r - r_n| \leq K^n \max |r - r_0|, \tag{1.1}$$

and

$$|r - r_n| \leq \frac{K^n}{1-K} \max |r_1 - r_0|, \text{ for all } n \geq 1. \tag{1.2}$$

Example 2. *Use the fixed point method to find the zero of the function $f(x) = x^3 - 3x^2 + 2$ in $[0,2]$, start with $r_0 = 1.5$.*

Solution: There are many possibilities to write $f(x) = 0$ as a fixed point form $x = g(x)$ using mathematical manipulations.

(1) $x = g_1(x) = x - x^3 + 3x^2 - 2.$

(2) $x = g_2(x) = \left(\frac{x^3+2}{3}\right)^{1/2}.$

(3) $x = g_3(x) = -\left(\frac{x^3+2}{3}\right)^{1/2}.$

(4) $x = g_4(x) = \left(\frac{2}{3-x}\right)^{1/2}.$

(5) $x = g_5(x) = \frac{-2}{x\left(x-3\right)}.$

(6) $x = g_6(x) = \left(3x^2 - 2\right)^{1/3}.$

(7) $x = g_7(x) = \left(3x - \frac{2}{x}\right)^{1/2}.$

For example, to obtain $g_1(x)$ just add x to both sides of the equation $-f(x) = 0$ and this is the simplest way to write the problem as a fixed point form

$$-f(x) = 0, \quad -x^3 + 3x^2 - 2 = 0, \quad \text{so} \quad x = x - x^3 + 3x^2 - 2 = g_1(x).$$

Also, $g_2(x)$ and $g_3(x)$ can be obtained as follows:

$$x^3 - 3x^2 + 2 = 0, \quad \text{so} \quad 3x^2 = x^3 + 2, \quad \text{and} \quad x^2 = \frac{x^3+2}{3},$$

implies that

$$x = \pm\left(\frac{x^3+2}{3}\right)^{1/2}, \quad \text{so} \quad g_2(x) = \left(\frac{x^3+2}{3}\right)^{1/2}, \text{ and } \quad g_3(x) = -\left(\frac{x^3+2}{3}\right)^{1/2}.$$

Note that it is important to check that the fixed point of each derived function g is a solution to the problem $f(x) = 0$. For example, because the solution is positive and lies between 0 and 2, so we choose the positive function $g_2(x)$, since the negative function $g_3(x)$ is not a choice here. The results are outlined in Tables 1.1 and 1.2 below.

n	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	$g_5(x)$
0	1.5	1.5	1.5	1.5	1.5
1	2.875	1.33853153	-1.33853153	1.15470054	0.88888889
2	1.90820313	1.21081272	$0 - 0.36i$	1.04107393	1.06578947
3	3.88369668	1.12177435		1.01042940	0.97018561
4	-11.44518863	1.06639827		1.00261759	1.01559099
5	$1.8788e+03$	1.03484519		1.00065504	0.99238449
6	$-6.6210e+09$	1.01787695		1.00016380	1.00385153
7	$2.9025e+29$	1.00905819		1.00004095	0.99808533
8		1.00455985		1.00001024	1.00096009
9		1.00228772		1.00000256	0.99952065
10		1.00114582		1.00000064	1.00023985
11		1.00057340		1.00000016	0.99988012
12				1.00000004	1.00005995
13				1.00000001	0.99997003
14				1.00000000	1.00001499
15					0.99999251
16					1.00000375
17					0.99999813
18					1.00000094
19					0.99999953
20					1.00000024
21					0.99999988

Table 1.1: Fixed Point Method Solution of Example 2

n	$g_6(x)$	$g_7(x)$
0	1.5	1.5
1	1.68098770	1.77951304
2	1.86406700	2.05295790
3	2.03474597	2.27698696
4	2.18422416	2.43979654
5	2.30913228	2.54944095
6	2.40992853	2.61989258
7	2.48919424	2.66388582
8	2.55035309	2.69088731
9	2.59688496	2.70728880
10	2.63192563	2.71718971
11	2.65811433	2.72314423
12	2.67757909	2.72671736
13	2.69198765	2.72885862
14	2.70262171	2.73014079
15	2.71045296	2.73090817
16	2.71621092	2.73136731
17	2.72043954	2.73164198
18	2.72354236	2.73180628
19	2.72581768	2.73190456
20	2.72748542	2.73196334
21	2.72870741	2.73199849

Table 1.2: Fixed Point Method Solution of Example 2

Remark 1.
- *The sequence of iterations* $\{r_n\}_{n=0}^{\infty}$ *generated by the fixed-point iteration rule* $r_{n+1} = g(r_n), n = 0, 1, \cdots$ *is either convergent or divergent.*
- *If the sequence of iterations is divergent, then we may have different types of divergence behaviour such as* ***monotone*** *or* ***oscillating*** *or* ***cyclic(repeated)***.
- *If the sequence of iterations is convergent, it may converge to another fixed point not the one we are interested in (may be it is not in the problem domain or domain of interest of the function* g*).*
- *If the sequence of iterations is convergent, the convergence may be* ***monotone*** *or* ***oscillating***.
- *Note that the Fixed Point Theorem does not explain what is the case if* $|g'(x)| = 1$. *In this case, the sequence of iterations also is either*

convergent or divergent and this depends on the closeness of the starting value r_0 to the fixed point r.

1.0.2 Newton's Method

Newton's method is also known as **Newton-Raphson method** is one of the most powerful and efficient numerical methods for *root-finding problems.* It is well-known and popular method and there are several variants and extensions of this method. There are more than one approach for deriving this method such as the *graphical technique* and *Taylor series technique*, and here we use both of them, we start with Taylor series approach. Let f, f', f'' are continuous functions on the interval $[a.b]$ (i.e. $f \in C^2[a,b]$). Let $r_0 \in [a,b]$ be an approximation to the zero r of the function f such that $f'(r_0) \neq 0$ and r_0 is "sufficiently close to r i.e. $|r - r_0|$ is relatively small". Let's start with first degree Taylor polynomial of $f(x)$ expanded about the initial approximation r_0 and compute it at $x = r$:

$$f(r) = f(r_0) + (r - r_0)f'(r_0) + \frac{(r - r_0)^2}{2} f''(\xi(r)),$$

where $\xi(r)$ lies between r_0 and r. Using the fact that $f(r) = 0$, this leads to

$$0 = f(r_0) + (r - r_0)f'(r_0) + \frac{(r - r_0)^2}{2} f''(\xi(r)).$$

Since $|r - r_0|$ is small then $(r - r_0)^2$ is much smaller, so we can neglect the third term in Taylor's expansion which contains this term (quadratic power term) to have

$$0 = f(r_0) + (r - r_0)f'(r_0).$$

Solving for r yields

$$r = r_0 - \frac{f(r_0)}{f'(r_0)}.$$

To, proceed, set $r = r_1$ in the Newton's formula to compute r_1 by using the known value r_0

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)},$$

and then we compute r_2 using the known value r_1

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)},$$

and by following the same fashion, we compute r_3, r_4 and so on. The general or *nth* form of Newton's method is:

$$r_n = r_{n-1} - \frac{f(r_{n-1})}{f'(r_{n-1})}, \quad n = 1, 2, \cdots . \tag{1.3}$$

This is called **Newton's formula** or **Newton-Raphson formula**.

The Graphical Approach: Let r_0 be an initial guess of the solution r of the equation $f(x) = 0$, then the curve of the function f crosses the x-axis at the point $(r, 0)$. The point $(r_0, f(r_0)$ lies on the curve near the point $(r, 0)$. The tangent line to the curve of f at the point $(r_0, f(r_0)$ intersects the x-axis at the point $(r_1, 0)$, then r_1 is the new approximation of the root and is closer to the root than r_0. The slope of the tangent line L joining the points $(r_0, f(r_0)$ and $(r_1, 0)$ is:

$$m = \frac{0 - f(r_0)}{r_1 - r_0}, \tag{1.4}$$

and also, we have

$$m = f'(r_0). \tag{1.5}$$

Equating the two slopes in (1.4) and (1.5), we get

$$f'(r_0) = \frac{-f(r_0)}{r_1 - r_0}, \tag{1.6}$$

solving for the new approximation r_1 we have

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)}. \tag{1.7}$$

By iterating (1.7), we obtain the Newton's formula in (1.3).

Theorem 4 (Convergence of the Newton's Method)**.** *Assume that the sequence of approximations* $\{r_n\}_{n=1}^{\infty}$ *of the root of the nonlinear equation* $f(x) = 0$ *produced by the Newton's iterative formula* (1.3) *converges to the root* r*. Then, if* r *is a simple root, the convergence is quadratic and the error bound is:*

$$|E_n| = |r_n - r_{n-1}| \approx \frac{f''(r)}{2f'(r)}|E_{n-1}|^2, \textit{ for sufficiently large } n. \tag{1.8}$$

If r is a multiple root of order $M > 1$, then the convergence is linear and the error bound is:

$$|E_n| = |r_n - r_{n-1}| \approx \frac{M-1}{M}|E_{n-1}|, \text{ for sufficiently large } n. \tag{1.9}$$

Note that the asymptotic error constants in the case of quadratic and linear convergence are $A = \frac{f''(r)}{2f'(r)}$ and $A = \frac{M-1}{M}$, respectively.

Example 3. *Use Newton's method to find the positive root accurate to within 10^{-5} for $f(x) = 3x - e^x = 0$. Start with the initial guess $r_0 = 1.5$.*

Solution: Start by finding the derivative of $f(x)$:

$f(x) = 3x - e^x,\ f'(x) = 3 - e^x,\ r_0 = 1.5, f(r_0) = 0.01831093, f'(r_0) = -1.48168907,$

so, the Newton-Raphson iteration formula for this problem is:

$$r_n = r_{n-1} - \frac{f(r_{n-1})}{f'(r_{n-1})} = r_{n-1} - \frac{3r_{n-1} - e^{r_{n-1}}}{3 - e^{r_{n-1}}},\ n = 1, 2, \cdots, .$$

Computing r_1 by using the known value r_0,

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)} = r_0 - \frac{3r_0 - e^{r_0}}{3 - e^{r_0}} = 1.5 - \frac{3(1.5) - e^{1.5}}{3 - e^{1.5}} = 1.51235815.$$

Now, compute $f(r_1)$ and $f'(r_1)$,

$$f(r_1) = 3r_1 - e^{r_1} = -0.00034364, \quad f'(r_1) = 3 - e^{r_1} = -1.53741808.$$

Next, we compute r_2,

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)} = r_1 - \frac{3r_1 - e^{r_1}}{3 - e^{r_1}} = 1.51213463,\ f(r_2) = -1.1e{-}07,\ f'(r_2) = -1.53640399.$$

A summary of the computations is given in Table 1.3.

n	r_n	$f(r_n)$	$f'(r_n)$
0	1.50000000	0.01831093	−1.48168907
1	1.51235815	−0.00034364	−1.53741808
2	1.51213463	−0.00000011	−1.53640399
3	1.51213455	−0.00000000	−1.53640365
4	1.51213455	0	−1.53640365

Table 1.3: Newton's Method Solution of Example 3

Remark 2. *One of the main drawbacks of the Newton's method is the possibility of division by zero when $f'(r_{n-1}) = 0$ in (1.3). In this case as a remedy we compute $f(r_{n-1})$ and if it is sufficiently close to zero, then we consider r_{n-1} is a reasonable approximation to the root r. Also, we have another problem when $f'(r_{n-1}) \approx 0$, i.e. when the tangent line to the curve of f at the point $(r_{n-1}, f(r_{n-1}))$ is nearly horizontal, then dividing by a very small number results in meaningless computations.*

Newton's Method for Finding the nth Roots

We start with square roots. Let $B > 0$ a real number and r_0 be an initial approximation to $\sqrt{B}$. Our goal is to find a square root of a number B. Let $x = \sqrt{B}$, so $x^2 = B$, which implies that $x^2 - B = 0$, define $f(x) = x^2 - B = 0$. Note that this equation has two roots $x = \pm\sqrt{B}$. Now, find the derivative of f, $f'(x) = 2x$ and use the Newton's fixed point formula

$$x = g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - B}{2x} = \frac{x^2 + B}{2x} = \frac{x + \frac{B}{x}}{2}.$$

Now, using Newton's iteration formula

$$r_{n+1} = \frac{r_n + \frac{B}{r_n}}{2}, \quad n = 0, 1, \cdots .$$

The sequence of iterations $\{r_n\}_{n=0}^{\infty}$ converges to $\sqrt{B}$. Note that in computing the square root of B, we do not need to evaluate f and f' and this makes the calculations easier and faster since we just need the values of the iterates r_n, $n = 0, 1, \cdots$.

Example 4. *Use Newton's square-root algorithm to find $\sqrt{3}$, use $r_0 = 1$.*

Solution: Starting with $r_0 = 1$ when $n = 0$, we have

$$r_1 = \frac{r_0 + \frac{3}{r_0}}{2} = \frac{1+3}{2} = 2.$$

For $n = 1$,

$$r_2 = \frac{r_1 + \frac{3}{r_1}}{2} = \frac{2 + \frac{3}{2}}{2} = 1.75.$$

$$n = 2,\ r_3 = \frac{r_2 + \frac{3}{r_2}}{2} = \frac{1.75 + \frac{3}{1.75}}{2} = 1.732142857142857.$$

$$n = 3,\ r_4 = \frac{r_3 + \frac{3}{r_3}}{2} = 1.732050810014727.$$

A summary of results is given in Table 1.4

n	r_n
0	1
1	2
2	1.75
3	1.732142857142857
4	1.732050810014727
5	1.732050807568877
6	1.732050807568877

Table 1.4: Newton's Method Solution of Example 4

1.0.3 Secant Method

Newton's method is a very powerful and efficient technique for solving root-finding problems but one of the drawbacks of the method is the need of derivative evaluations of f at the approximations $r_n, n \geq 0$, and this is not a trivial task. To avoid this we introduce **secant method** which is a variation of Newton's method. Secant method is similar to the false position method but it differs in the way of choosing the succeeding terms.We start with two initial points $(r_0, f(r_0))$ and $(r_1, f(r_1))$ near the point $(r, 0)$, where r is the root of equation $f(x) = 0$. Define the point $(r_2, 0)$ to be the point of intersection of the secant line joining the points $(r_0, f(r_0))$ and $(r_1, f(r_1))$ with the x-axis. Geometrically, the abscissa of the point of intersection r_2 is closer to the root r than to either r_0 and r_1.

The slope of the secant line relating these three points $(r_0, f(r_0))$, $(r_1, f(r_1))$ and $(r_2, f(r_2))$ is:

$$m = \frac{f(r_1) - f(r_0)}{r_1 - r_0} \quad \text{and} \quad m = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = \frac{0 - f(r_1)}{r_2 - r_1} = \frac{-f(r_1)}{r_2 - r_1}.$$

Equating the two values of the slope, we have

$$\frac{f(r_1) - f(r_0)}{r_1 - r_0} = \frac{-f(r_1)}{r_2 - r_1}.$$

Solving slope's equation for r_2, we obtain

$$r_2 = r_1 - \frac{f(r_1)(r_1 - r_0)}{f(r_1) - f(r_0)}.$$

So, the general form of the secant method is:

$$r_{n+2} = r_{n+1} - \frac{f(r_{n+1})(r_{n+1} - r_n)}{f(r_{n+1}) - f(r_n)}, \ n = 0, 1, \cdots, .$$

Example 5. *Find the root of equation $x - \cos(x) = 0$ using the secant method and the two initial guesses $r_0 = 0.5$ and $r_1 = 0.6$.*

Solution: To compute the first approximation r_2, we need to compute $f(r_0)$ and $f(r_1)$

$$f(r_0) = f(0.5) = 0.5 - \cos(0.5) = -0.377582560000000,$$

$$f(r_1) = f(0.6) = 0.6 - \cos(0.6) = -0.225335610000000.$$

So,

$$r_2 = r_1 - \frac{f(r_1)(r_1 - r_0)}{f(r_1) - f(r_0)} = 0.6 - \frac{f(0.6)(0.6 - 0.5)}{f(0.6) - f(0.5)} = 0.748006655882730,$$

$$f(r_2) = r_2 - \cos(r_2) = 0.014960500949714.$$

Now, we compute the next approximation r_3,

$$r_3 = r_2 - \frac{f(r_2)(r_2 - r_1)}{f(r_2) - f(r_1)} = 0.738791967963291, \; f(r_3) = -0.000490613128583.$$

Continuing until satisfying the required accuracy. A summary of the calculations is given in Table 1.5.

n	r_n	$f(r_n)$
0	0.500000000000000	−0.377582560000000
1	0.600000000000000	−0.225335610000000
2	0.748006655882730	0.014960500949714
3	0.738791967963291	−0.000490613128583
4	0.739084558312839	−0.000000962163319
5	0.739085133252381	0.000000000062293
6	0.739085133215161	0
7	0.739085133215161	0

Table 1.5: Secant Method Solution of Example 5

1.1 Acceleration of Iterative Methods

The linear convergence of a sequence $\{r_n\}$ to the limit r such as the sequences of the fixed point iterations can be accelerated by using some techniques such as **Aitken's Δ^2 method (Aitken's acceleration)** and **Steffensen's method**.

1.1.1 Modified Newton's Methods

Newton's method is a fixed point method since it can be written as

$$x = g(x) = x - \frac{f(x)}{f'(x)},$$

and in iterative way

$$r_n = g(r_{n-1}) = r_{n-1} - \frac{f(r_{n-1})}{f'(r_{n-1})}, \quad n = 1, 2, \cdots,$$

and this is called **Newton-Raphson iteration formula** or simply **Newton's iteration**. The convergence of Newton's method can be modified to accelerate its rate of convergence at the root $x = r$ of order $M > 1$

$$r_n = r_{n-1} - \frac{f(r_{n-1})f'(r_{n-1})}{(f'(r_{n-1}))^2 - f(r_{n-1})f''(r_{n-1})}, \quad n = 1, 2, \cdots.$$

This formula is called a **modified Newton's method**.

Also, Newton's method can be accelerated in an another way.

Theorem 5 (Acceleration of Newton's Iteration)**.** *Assume that Newton's method produces a linearly convergent sequence to the root $x = r$ of order $M > 1$. Then Newton's iteration formula*

$$r_n = r_{n-1} - \frac{Mf(r_{n-1})}{f'(r_{n-1})}, \quad n = 1, 2, \cdots,$$

produces a quadratically convergent sequence $\{r_n\}_{n=0}^{\infty}$ to the root $x = r$.

Example 6. *Show that $r = 1$ is a double zero (double root) of $f(x) = -x^3 + 3x - 2 = 0$. Start with $r_0 = 1.25$ as an initial guess of r and compare the performance of Newton's method and accelerated Newton's method for solving $f(x) = 0$.*

Solution : Since $r = 1$ is a double root then $M = 2$, so the accelerated Newton's method becomes

$$r_n = r_{n-1} - \frac{2f(r_{n-1})}{f'(r_{n-1})} = r_{n-1} - \frac{2(-r_{n-1}^3 + 3r_{n-1} - 2)}{-3r_{n-1}^2 + 3}, \quad n = 1, 2, \cdots,$$

or

$$r_n = r_{n-1} - \frac{-2r_{n-1}^3 + 6r_{n-1} - 4}{-3r_{n-1}^2 + 3}, \quad n = 1, 2, \cdots.$$

Start by computing r_1

$$r_1 = r_0 - \frac{-2r_0^3 + 6r_0 - 4}{-3r_0^2 + 3} = 1.25 - \frac{-2(1.25)^3 + 6(1.25) - 4}{-3(1.25)^2 + 3} = 1.00925926.$$

Table 1.6 compares the performance of both methods.

n	Newton's Method	Accelerated Newton's Method
0	1.25	1.25
1	1.12962963	1.00925926
2	1.06612990	1.00001422
3	1.03341772	1.00001422
4	1.01680039	1.00000000
5	1.00842352	
6	1.00421765	
7	1.00211030	
8	1.00105552	
9	1.00052785	
10	1.00026395	
11	1.00013198	
12	1.00006599	
13	1.00003300	
14	1.00001650	
15	1.00000825	
16	1.00000413	
17	1.00000207	
18	1.00000104	
19	1.00000052	
20	1.00000026	
21	1.00000013	
22	1.00000007	
23	1.00000004	
24	1.00000002	
24	1.00000001	
25	1.00000001	

Table 1.6: Newton's and Accelerated Newton's Methods Solutions of Example 6

1.2 Computing Roots of Polynomials

Computing roots of polynomials has important applications in different areas of mathematics and other sciences.

Definition 3 (*nth* Degree Polynomial)**.** *A polynomial of degree n has the general form*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients a_i, $i = 0, 1, \cdots, n$, are real numbers (constants) and $a_n \neq 0$. The nth degree polynomial $P(x)$ is sometimes referred to as $P_n(x)$, and also named algebraic polynomial.

Note that the zero function $P(x) = 0$ is a polynomial but has no degree.There are several techniques for finding zeros of polynomials in the literature such as **Müller method**, **Laguerre's method**, **Bairstow method**, **Brent's method** and **Jenkins-Traub method**, and these methods are beyond the scope of this lecture notes and interested readers can see the references.

Remark 3 (**Hybrid Methods**)**.** *The global methods are guaranteed to converge to the root of the problem if given an initial interval such that the function changes sign on this interval, but these methods are slow and have linear convergence rates. The local methods are faster but it is not guaranteed to converge to the root unless we start sufficiently close to the root, and these methods have higher order convergence rates. Hence, to make balance between the good features in both methods, there are some methods start few steps with closed methods to guarantee the convergence and then move to open methods to speed up the convergence, these methods are called the* ***hybrid methods****.*

Exercises

Exercise 6. *Find the solution to the equation* $e^x - x - 1 = 0$ *accurate to six decimal places (i.e.* $\epsilon = 0.0000001$*) using Newton's and modified Newton's methods. Start with* $r_0 = 0.6$*. Compare the results of both methods.*

Exercise 7. *Use the secant method to find the solution accurate to within* 10^{-5} *to the following problem* $x\sin(x) - 1 = 0, \quad 0 \le x \le 2$.

Exercise 8. *Use the fixed point method to locate the root of* $f(x) = x - e^{-x} = 0$*, start with an initial guess of* $x = 0.1$.

Exercise 9. *Let* $f(x) = x^2 - 5$ *and* $r_0 = 1.5$*. Use secant, fixed point, Newton's and modified Newton's methods to find* r_7 *the approximation to the positive root* $r = \sqrt{5}$.

Exercise 10. *Use modified (accelerated) Newton's method to solve the equation* $x^2 - 3x - 1 = 0$ *in the interval* $[-1, 1]$.