Solving Systems of Linear Equations

Lecture Notes

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0.1 Introduction

Many phenomena and relationships in nature and real life applications are linear, meaning that results and their causes are proportional to each other. Solving linear algebraic equations is a topic of great importance in numerical analysis and other scientific disciplines such as engineering and physics. Solutions to Many problems reduced to solve a system of linear equations. For example, in finite element analysis a solution of a partial differential equation is reduced to solve a system of linear equations.

0.2 Norms of Matrices and Vectors

In error and convergence analyses we need a measure to determine the distance (difference) between the exact solution and approximate solution or to determine the differences between consecutive approximations.

Definition 1 (Vector Norm)**.** *A* ***vector norm*** *is a real-valued function* $||.|| : \mathbb{R}^n \to \mathbb{R}$ *satisfies the following conditions:*

(i) $\|\mathbf{x}\| \geq 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(ii) $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ *for all* $\alpha \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *(Triangle Inequality).*

Definition 2 (l_1 Vector Norm). *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the* l_1 ***norm*** *for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

Definition 3 (Euclidean Vector Norm). *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the* ***Euclidean norm*** *(*l_2 ***norm****) for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}.$$

Definition 4 (Maximum Vector Norm). *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the* ***maximum norm*** *(*l_∞ ***norm****) for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}.$$

Remark 1. *Note that when* $n = 1$ *both norms reduce to the absolute value function of real numbers.*

Example 1. *Determine the* l_1 *norm,* l_2 *norm and* l_∞ *norm of the vector* $\mathbf{x} = (1, 0, -1, 2, 3)'$.

Solution: The required norms of vector $\mathbf{x} = (1, 0, -1, 2, 3)'$ in $\mathbb{R}^5$ are:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{5} |x_i| = |x_1| + |x_2| + |x_3| + |x_4| + |x_5| = |1| + |0| + |-1| + |2| + |3| = 7,$$

$$\begin{aligned} \|\mathbf{x}\|_2 &= \Big(\sum_{i=1}^{5} x_i^2\Big)^{1/2} = \Big(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\Big)^{1/2} \\ &= \Big((1)^2 + (0)^2 + (-1)^2 + (2)^2 + (3)^2\Big)^{1/2} = \Big(15\Big)^{1/2}, \end{aligned}$$

and

$$\begin{aligned}\|\mathbf{x}\|_\infty &= \max_{1\le i\le 5}\{|x_i|\} = \max\{|x_1|,|x_2|,|x_3|,|x_4|,|x_5|\}\\ &= \max\{|1|,|0|,|-1|,|2|,|3|\} = 3.\end{aligned}$$

Definition 5 (Matrix Norm). *A* ***matrix norm*** *is a real-valued function* $\|.\| : \mathbb{R}^{n\times m} \to \mathbb{R}$ *satisfies the following conditions:*

(i) $\|A\| \ge 0$ *for all* $A \in \mathbb{R}^{n\times m}$*.*

(ii) $\|A\| = 0$ *if and only if* $A = \mathbf{0}$ *for all* $A \in \mathbb{R}^{n\times m}$*.*

(iii) $\|\alpha A\| = |\alpha|\|A\|$ *for all* $\alpha \in \mathbb{R}$ *and* $A \in \mathbb{R}^{n\times m}$*.*

(iv) $\|A + B\| \le \|A\| + \|B\|$ *for all* $A, B \in \mathbb{R}^{n\times m}$ *(Triangle Inequality).*

If matrix norm is related to a vector norm, then we have two additional properties:

(v) $\|AB\| \le \|A\|\|B\|$ *for all* $A, B \in \mathbb{R}^{n\times m}$*.*

(vi) $\|A\mathbf{x}\| \le \|A\|\|\mathbf{x}\|$ *for all* $A \in \mathbb{R}^{n\times m}$ *and* $\mathbf{x} \in \mathbb{R}^n$*.*

We give here some equivalent definitions of the matrix norm particularly when matrix norm is related to the vector norm.

Definition 6 (Subordinate Matrix Norm). *Let* A *is a* $n \times n$ *matrix and* $\mathbf{x} \in \mathbb{R}^n$*, then the* ***subordinate matrix norm*** *is defined by*

$$\|A\| = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n \mathit{and}\ \|\mathbf{x}\| = 1\}.$$

or, alternatively

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Definition 7 (Natural Matrix Norm). *Let* A *is a* $n \times n$ *matrix and for any* $\mathbf{z} \neq \mathbf{0}$*, and* $\mathbf{x} = \frac{\mathbf{z}}{\|\mathbf{z}\|}$ *is the unit vector. Then the* ***natural / reduced matrix norm*** *is defined by*

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\mathbf{z}\neq 0} \left\| A\Big(\frac{\mathbf{z}}{\|\mathbf{z}\|}\Big)\right\| = \max_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|},$$

or, alternatively

$$\|A\| = \max_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}.$$

Definition 8 (l_1 Matrix Norm)**.** *Let A is a $n\times n$ matrix and $\mathbf{x}=(x_1,x_2,\cdots,x_n)'$. Then the l_1 **matrix norm** is defined by*

$$\|A\|_1=\max_{\|\mathbf{x}\|_1=1}\|A\mathbf{x}\|_1=\max_{1\le i\le n}\sum_{i=1}^{n}|a_{ij}|.$$

Definition 9 (Spectral Matrix Norm)**.** *Let A is a $n\times n$ matrix and $\mathbf{x}=(x_1,x_2,\cdots,x_n)'$. Then the **spectral / l_2-matrix norm** is defined by*

$$\|A\|_2=\max_{\|\mathbf{x}\|_2=1}\|A\mathbf{x}\|_2=\max_{1\le i\le n}\sqrt{|\sigma_{\max}|},$$

*where σ_i are the eigenvalues of A^TA, which are called the **singular values** of A and the largest eigenvalue in absolute value ($|\sigma_{\max}|$) is called the **spectral radius** of A.*

Definition 10 (l_∞ Matrix Norm)**.** *Let A is a $n\times n$ matrix and $\mathbf{x}=(x_1,x_2,\cdots,x_n)'$. Then the l_∞ **(maximum)matrix norm** is defined by*

$$\|A\|_\infty=\max_{\|\mathbf{x}\|_\infty=1}\|A\mathbf{x}\|_\infty=\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|.$$

Remark 2. *Note that $\|I\|=1$.*

Example 2. *Determine $\|A\|_\infty$ for the matrix*

$$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 5 & 3\\ -1 & 6 & -4\end{bmatrix}.$$

Solution: For $i=1$, we have

$$\sum_{j=1}^{3}|a_{1j}|=|a_{11}|+|a_{12}|+|a_{13}|=|1|+|-1|+|2|=4,$$

and for $i=2$, we obtain

$$\sum_{j=1}^{3}|a_{2j}|=|a_{21}|+|a_{22}|+|a_{23}|=|0|+|5|+|3|=8,$$

for $i=3$, we get

$$\sum_{j=1}^{3}|a_{3j}|=|a_{31}|+|a_{32}|+|a_{33}|=|-1|+|6|+|-4|=11.$$

Consequently,

$$\|A\|_\infty = \max_{1\le i\le 3} \sum_{j=1}^{3} |a_{ij}| = \max\{4, 8, 11\} = 11.$$

0.3 Direct Methods

Direct methods are techniques used for solving and obtaining the exact solutions (in theory) of linear algebraic equations in a finite number of steps. The main widely used direct methods are **Gaussian elimination method** and **Gauss-Jordan method**.

Consider the following linear system of dimension $n \times (n+1)$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

This system can be written in concise form by using matrix notation as $AX = B$ as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where $A_{n\times n}$ is square matrix and is called a **coefficient matrix**, $B_{n\times 1}$ is a column vector known as the **right hand side vector** and $X_{n\times 1}$ is a column vector known as **unknowns vector**. Also, this system can be written as

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],$$

where $[A|B]$ is called the **augmented matrix**.

0.3.1 Backward Substitution Method

Backward substitution also called **back substitution** is an algorithm or technique used for solving **upper-triangular systems** which are systems such that their coefficient matrices are upper-triangular matrices. Assume that we have the following upper-triangular system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n-1}x_{n-1} + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n-1}x_{n-1} + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \cdots + a_{3n-1}x_{n-1} + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{n-1n-1}x_{n-1} + a_{n-1n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n.
\end{aligned}$$

To find a solution to this system we follow the following steps provided that $x_{rr} \neq 0,\ r = 1, 2, \cdots, n$:

(1) Solve the last (nth) equation for x_n:

$$x_n = \frac{b_n}{a_{nn}}.$$

(2) Substitute x_n in the next-to-last $((n-1)th)$ equation and solve it for x_{n-1}:

$$x_{n-1} = \frac{b_{n-1} - a_{n-1n}x_n}{a_{n-1n-1}}.$$

(3) Now, x_n and x_{n-1} are known and can be used to find x_{n-2}:

$$x_{n-2} = \frac{b_{n-2} - a_{n-2n-1}x_{n-1} - a_{n-2n}x_n}{a_{n-2n-2}}.$$

(4) Continuing in this way until we arrive at the general step:

$$x_r = \frac{b_r - \sum_{j=r+1}^{n} a_{rj}x_j}{a_{rr}}, \quad r = n-1, n-2, \cdots 1.$$

Example 3. *Solve the following linear system using back substitution method*

$$\begin{aligned}
3x_1 + 2x_2 - x_3 + x_4 &= 10 \\
x_2 - x_3 + 2x_4 &= 9 \\
3x_3 - x_4 &= 1 \\
3x_4 &= 6
\end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1 + 2}{3} = \frac{3}{3} = 1.$$

Now, use values $x_3 = 1$ and $x_4 = 2$ in the second equation to find x_2

$$x_2 = 9 + x_3 - 2x_4 = 9 + 1 - 4 = 6.$$

Finally, solve the first equation for x_1 yields

$$x_1 = \frac{10 - 2x_2 + x_3 - x_4}{3} = \frac{10 - 12 + 1 - 2}{3} = \frac{-3}{3} = -1.$$

Example 4. *Show that the following linear system has no solution*

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= 10 \\ 0x_2 - x_3 + 2x_4 &= 9 \\ 3x_3 - x_4 &= 1 \\ 3x_4 &= 6 \end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1 + 2}{3} = \frac{3}{3} = 1.$$

Also, from the second equation we have

$$x_3 = 9 - 2x_4 = 9 - 4 = 5.$$

This contradiction implies that the linear system in above has no solution.

Example 5. *Show that the following linear system has infinitely many solutions*

$$\begin{aligned} 3x_1 + 3x_2 - x_3 + x_4 &= 10 \\ 0x_2 + x_3 + 0x_4 &= 1 \\ 3x_3 - x_4 &= 1 \\ 3x_4 &= 6 \end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1+2}{3} = \frac{3}{3} = 1.$$

Also, from the second equation we have

$$x_3 = 1.$$

Solve the first equation for x_2 yields

$$x_2 = \frac{10 - 3x_1 + x_3 - x_4}{3} = \frac{10 - 3x_1 + 1 - 2}{3} = \frac{9 - 3x_1}{3} = 3 - x_1.$$

Note that the equation for x_2 has infinitely many solutions since it depends upon x_1 which takes infinitely many values. Now, let $x_1 = 1$, we have $x_2 = 2$. Hence the solution set of the system is:

$$x_1 = 1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 2.$$

0.3.2 Forward Substitution Method

Forward substitution is an algorithm or technique used for solving **lower-triangular systems** which are systems such that their coefficient matrices are lower-triangular matrices.

$$\begin{aligned}
&a_{11}x_1 &&= b_1\\
&a_{21}x_1 + a_{22}x_2 &&= b_2\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &&= b_3\\
& &&\vdots\\
&a_{n-11}x_1 + a_{n-12}x_2 + a_{n-13}x_3 + \cdots + a_{n-1n-1}x_{n-1} &&= b_{n-1}\\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn-1}x_{n-1} + a_{nn}x_n &&= b_n.
\end{aligned}$$

To find a solution to this system we follow the following steps provided that $x_{rr} \neq 0,\ r = 1, 2, \cdots, n$:

(1) Solve the first ($1st$) equation for x_1:

$$x_1 = \frac{b_1}{a_{11}}.$$

(2) Substitute x_1 in the second equation ($2nd$) equation and solve it for x_2:

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}.$$

(3) Now, x_1 and x_2 are known and can be used to find x_3:

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}.$$

(4) Continuing in this way until we arrive at the general step:

$$x_r = \frac{b_r - \sum_{j=1}^{r-1} a_{rj}x_j}{a_{rr}}, \quad r = 2, 3, \cdots n.$$

Example 6. *Use the forward substitution method for solving the following linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + 2x_2 &&= -1 \\ &x_1 - x_2 + 5x_3 &&= 0.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

Use x_1 and x_2 to find x_3

$$x_3 = \frac{0.5 - x_1 + x_2}{5} = \frac{0.5 - 2 - 2.5}{5} = \frac{-4}{5} = -0.8.$$

Finally, solve for x_4 to have

$$x_4 = \frac{2 - 0.1x_1 - 2x_2 + x_3}{2} = \frac{2 - 0.1(2) - 2(-2.5) - 0.8}{2} = \frac{6}{2} = 3.$$

Example 7. *Show that there is no solution to the linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + 2x_2 &&= -1 \\ &x_1 - x_2 + 0x_3 &&= 0.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 in the second equation to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

From the third equation we have

$$x_2 = x_1 - 0.5 = 2 - 0.5 = 1.5.$$

This contradiction indicates that there is no solution to the system in above.

Example 8. *Show that there are infinitely many solution to the following linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + 2x_2 &&= -1 \\ &0x_1 - x_2 + 0x_3 &&= 2.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 in the second equation to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

From the third equation we have

$$x_2 = -2.5.$$

Solving the last equation for x_3 we obtain

$$x_3 = -2 + 0.1x_1 + 2x_2 + 2x_4 = -2 + 0.1(2) + 2(-2.5) + 2x_4 = 2x_4 - 6.8,$$

which has infinitely many solutions. Hence, the above linear system has infinitely many solutions. If we choose $x_4 = 8$, then we get $x_3 = 10.8$. So, the solution set is:

$$x_1 = 2,\ x_2 = -2.5,\ x_3 = 10.8,\ x_4 = 8.$$

0.3.3 Gaussian Elimination Method

Gaussian elimination method is also known as **Gauss elimination method** or simply **elimination method**. It is a direct method used for solving a system of linear algebraic equations. In this method we transform the linear system to an equivalent upper or lower triangular system and then solve it by backward or forward substitution. The process of transforming the linear system to an equivalent upper or lower triangular system is called **trianguarisation**.

Definition 11 (Equivalent Systems). *Two linear algebraic systems of dimension $n \times n$ is said to be they are **equivalent** if they have the same solution sets.*

Definition 12 (Elementary Transformations). *The following operations performed on a linear system transform it to an equivalent system:*

- ***Interchanges:*** *Changing the order of any two equations in the system.*
- ***Replacement:*** *Any equation of the system can be replaced by itself and a nonzero multiple of any other equation in the system.*
- ***Scaling:*** *Multiplying any equation in the system by a nonzero real constant.*

Definition 13 (Elementary Row Operations). *The following operations performed on a linear system transform it to an equivalent system:*

- ***Interchanges:*** *Changing the order of any two rows in the matrix.*
- ***Replacement:*** *Any row in the matrix can be replaced by its sum and a nonzero multiple of any other row in the matrix.*
- ***Scaling:*** *Multiplying any row in the matrix by a nonzero real constant.*

Pivoting

Pivoting is an important process used in solving linear systems in conjunction with Gaussian elimination and there are different types of pivoting strategies as outlined below:

1. **Trivial Pivoting:** The process of using the element (entry) a_{kk} in the coefficient matrix A to eliminate the entries $a_{rk},\ r = k+1, k+2, \cdots n$ is called **pivoting process**. The element a_{kk} is called **pivotal element** and the kth row is called **pivotal row**. If the entry $a_{kk} = 0$, then the row k cannot be used to eliminate the entries a_{rk}, $r = k+1, k+2, \cdots, n$ and we need to find a row r such that $a_{rk} \neq 0$, $r > k$, and then interchange the row k and the row r such that the pivotal element is nonzero. This process is called the **trivial pivoting**, also, if no interchange or switching between the rows is performed then the process is called only **pivoting** or **trivial pivoting**.

2. **Partial Pivoting:** To reduce the round-off errors or propagation of errors it is advisable to search for the the greatest element in the magnitude in column r that lies on or below the main diagonal, and then move it to the main diagonal in the pivotal row r to be the pivotal element and use it to eliminate the entries in the column r below the main diagonal, this process is called the **partial pivoting**. Determine row k below the main diagonal in which there is the largest element in the absolute value as follows:

$$a_{kr} = \max\{|a_{rr}|, |a_{r+1r}|, \cdots, |a_{n-1r}|, |a_{nr}|\}, \tag{1}$$

and then interchange the row k and row r for $k > r$. Now, since the entry in the main diagonal has the larges absolute value then the values of all the multipliers are:

$$|m_{kr}| \leq 1, k = r+1, r+2, \cdots, n,$$

and this will be helpful to keep the magnitudes of elements in the current matrix are relatively the same magnitudes of the elements in the original coefficient matrix.

3. **Scaled Pivoting:** In this approach, the pivoting element is chosen to be the largest in magnitude relative to the elements which lie in the same row. This type of pivoting is used when the entries in the same row vary largely in magnitude.

4. **Complete Pivoting:** In this technique, we use both partial and scaled pivoting and is sometimes referred to as **scaled partial pivoting** or **equilibrating**. In this process, we search all the entries in the column r that lie on or below the main diagonal for the largest entry in the magnitude relative to the entries in its row. Hence, We interchange both the columns and rows to find the largest entry in absolute value, i.e. we searching for largest entry in the matrix and for this reason this type of pivoting is also known as **maximal pivoting**. we start the process by searching all the rows r to n for the largest entry in absolute value in each row, we denote this element by p_k:

$$p_k = \max\{|a_{kr}|, |a_{kr+1}|, \cdots, |a_{kn-1}|, |a_{kn}|\},\ k = r, r+1, \cdots, n. \quad (2)$$

Then, to locate the pivoting row, we need to compute

$$\frac{a_{kr}}{p_k} = \max\{|\frac{a_{rr}}{p_r}|, |\frac{a_{r+1r}}{p_{r+1}}|, \cdots, |\frac{a_{n-1r}}{p_{n-1}}|, |\frac{a_{nr}}{p_n}|\}. \quad (3)$$

Then, interchange the row r and k, except the case when $r = k$.

Example 9. *Write the following linear system in the augmented form and then solve it by using Gauss elimination method with trivial pivoting.*

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 12 \\ 2x_1 + x_2 + x_3 + x_4 &= 10 \\ -3x_1 - x_2 + 4x_3 + x_4 &= 2 \\ x_1 + x_2 - x_3 + 3x_4 &= 6 \end{aligned}$$

.

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 2 & 1 & 1 & 1 & 10 \\ -3 & -1 & 4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right]$$

The first row is the pivotal row, so the pivotal element is $a_{11} = 1$ and is used to eliminate the first column below the diagonal. We will denote by m_{r1} to the multiples of the row 1 subtracted from row r for $r = 2, 3, 4$. Multiplying the first row by $m_{21} = -2$ and add it to the second row to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ -3 & -1 & 4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right].$$

Now, multiply the first row by $m_{31} = 3$ and add it to the third row to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 5 & 1 & 13 & 38 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right].$$

Multiplying the first row by $m_{41} = -1$ and adding it to the fourth row yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 5 & 1 & 13 & 38 \\ 0 & -1 & 0 & -1 & -6 \end{array}\right].$$

Now, the pivotal row is the second row and the pivotal element is $a_{22} = -3$. Multiply the second row by $m_{32} = \frac{5}{3}$ to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & -1 & 0 & -1 & -6 \end{array}\right].$$

Multiply the the second row by $m_{42} = \frac{-1}{3}$ and add it to the fourth row to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & 0 & -1 & 4/3 & -4/3 \end{array}\right].$$

Now, the pivotal row is the third row and the third element is $a_{33} = 6$. Finally, multiply the third row by $m_{43} = \frac{1}{6}$ to the fourth row to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & 0 & 0 & 14/9 & 10/9 \end{array}\right].$$

Now, note that the coefficient matrix is transformed into an upper triangular matrix and can be solved by backward substitution method. Firstly, we from the last row we compute

$$x_4 = \frac{10/9}{14/9} = \frac{5}{7}.$$

Use the third row to solve for x_3

$$x_3 = \frac{44/3 - 4/3(5/7)}{6} = \frac{288/21}{6} = \frac{16}{7}.$$

Now, solve the second equation for x_2

$$x_2 = \frac{-14 - 3x_3 + 7x_4}{-3} = \frac{-14 - 3(16/7) + 7(5/7)}{-3} = \frac{111}{21} = \frac{37}{7}.$$

Finally, solve the first equation for x_1

$$x_1 = 12 - 2x_2 + x_3 - 4x_4 = 12 - 2(37/7) + 16/7 - 4(5/7) = \frac{6}{7}.$$

Example 10. *Solve the following linear system using Gauss elimination method by using forward substitution technique*

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6 \end{aligned}$$

.

Solution: We start our solution strategy by transforming this square system to equivalent lower-triangular system and then solve it by using forward substitution method. Write the system in augmented matrix form

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

$$\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ R_1 + 2R_2 \\ g \\ h \\ i \\ j \end{array}$$

Note that now the pivotal row is the fourth row and the pivotal element is $a_{44} = 2$. Multiply the fourth row by the multiple $m_{14} = -2$ and it to the first row to have

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Multiply the fourth row by $m_{24} = \frac{-3}{2}$ and add it to the second row to obtain

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 13/2 & -3/2 & -1/2 & 0 & 19 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now multiply the fourth equation by $m_{34} = \frac{-1}{2}$ and add it to the third row to have

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 13/2 & -3/2 & -1/2 & 0 & 19 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

The pivotal row now is the third row and the pivotal element is $a_{33} = 1/2$. Add the third row to the second row (i.e. multiply it by $m_{23} = 1$) to get

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 12 & 0 & 0 & 0 & 36 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now

$$\left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & 36 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ 7 & 0 & -5 & 0 & 1 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

The pivotal row (third row) is used to eliminate elements in the second row and the pivotal element is $a_{33} = -5$. Multiply the third row by $m_{23} = \frac{1}{10}$ to have

$$\left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & 36 \\ 31/5 & 3/2 & 0 & 0 & 171/10 \\ 7 & 0 & -5 & 0 & 1 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now, use forward substitution to solve the lower-triangular matrix. solve the first equation for x_1

$$x_1 = \frac{36}{12} = 3.$$

Use the equation to find x_2

$$x_2 = \frac{171/10 - (31/5)3}{3/2} = -1.$$

Now, solve the third equation for x_3

$$x_3 = \frac{1 - 7(3)}{-5} = 4.$$

Finally, solve the fourth equation for x_4

$$x_4 = \frac{6 - (-3)(3) - 1(-1) - 3(4)}{2} = 2.$$

0.3.4 Gauss-Jordan Elimination Method

In this method instead of transforming the coefficient matrix into upper or lower triangular system, we transform the coefficient matrix into diagonal (in particular identity) matrix using elementary row operations.

Example 11. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} 3x_1 + 4x_2 + 3x_3 &= 10 \\ x_1 + 5x_2 - x_3 &= 7 \\ 6x_1 + 3x_2 + 7x_3 &= 15 \end{aligned}$$

.

Solution: Express the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 3 & 4 & 3 & 10 \\ 1 & 5 & -1 & 7 \\ 6 & 3 & 7 & 15 \end{array}\right].$$

The pivot row is the first row and the pivot element is $a_{11} = 3$. Multiply it by $m_{11} = 1/3$ to get

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 1 & 5 & -1 & 7 \\ 6 & 3 & 7 & 15 \end{array}\right].$$

Subtract the second equation from the first (i.e. multiply it by $m_{21} = -1$) and multiply the fist equation by $m_{31} = -6$ and add it to the third equation to have

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 0 & -11/3 & 2 & -11/3 \\ 0 & -5 & 1 & -5 \end{array}\right].$$

Now, the pivot row is the second row and the pivot element $a_{22} = -11/3$. Multiply it by $m_{22} = -3/11$ to have

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 0 & 1 & -6/11 & 1 \\ 0 & -5 & 1 & -5 \end{array}\right].$$

Multiply the first and third rows by $m_{12} = -4/3$ and $m_{32} = 5$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/11 & 2 \\ 0 & 1 & -6/11 & 1 \\ 0 & 0 & -19/11 & 0 \end{array}\right].$$

The pivot element now is third row and the pivot element is $a_{33} = -19/11$. Multiply it by $m_{33} = -11/19$ to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/11 & 2 \\ 0 & 1 & -6/11 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Finally, multiply the third row by $m_{13} = -19/11$ and $m_{23} = 6/11$ and add it to the first and second rows to have

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence, we have $x_1 = 2$, $x_2 = 1$ and $x_3 = 0$.

Example 12. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} -2x_1 + x_2 + 5x_3 &= 15 \\ 4x_1 - 8x_2 + x_3 &= -21 \\ 4x_1 - x_2 + x_3 &= 7 \end{aligned}$$

.

Solution: Write the system in augmented matrix form

$$\left[\begin{array}{ccc|c} -2 & 1 & 5 & 15 \\ 4 & -8 & 1 & -21 \\ 4 & -1 & 1 & 7 \end{array}\right].$$

Multiply the first row by $m_{21} = m_{31} = -2$ and it to the second and third rows respectively, to obtain

$$\left[\begin{array}{ccc|c} -2 & 1 & 5 & 15 \\ 0 & -6 & 11 & 9 \\ 0 & 1 & 11 & 37 \end{array}\right].$$

Now, multiply the second row by $m_{12} = m_{32} = \frac{1}{6}$ and it to the first and third rows respectively, to have

$$\left[\begin{array}{ccc|c} -2 & 0 & 41/6 & 33/2 \\ 0 & -6 & 11 & 9 \\ 0 & 0 & 77/6 & 77/2 \end{array}\right].$$

Finally, multiply the third row by $m_{13} = \frac{-41}{77}$ and $m_{32} = \frac{-6}{7}$ and it to the first and third rows respectively, to obtain

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & -4 \\ 0 & -6 & 0 & -24 \\ 0 & 0 & 77/6 & 77/2 \end{array}\right],$$

implies that

$$x_1 = \frac{-4}{-2} = 2, \quad x_2 = \frac{-24}{-6} = 4 \quad \text{and} \quad x_3 = \frac{77/2}{77/6} = 3.$$

0.4 LU and Cholesky Factorisations

In this section we will discuss the triangular factorisations of matrices.

Definition 14 (Positive Definite Matrix)**.** *Let $A_{n\times n}$ be symmetric real matrix and $\mathbf{x} \in \mathbb{R}^n$ a nonzero vector. Then, A is said to be* ***positive definite matrix*** *if $A = A^T$ and $\mathbf{x}^T A\mathbf{x} > 0$ for any $\mathbf{x}$.*

Remark 3. *Note that the matrix A is nonsingular by definition.*

Definition 15 (Triangular Factorisation)**.** *Assume that A is a nonsingular matrix. It said to be A has a* ***triangular factorisation*** *or* ***triangular decomposition*** *if it can be factorised as a product of unit lower-triangular matrix L and an upper triangular matrix U:*

$$A = LU.$$

or in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Note that since A is nonsingular matrix this implies that $u_{rr} \neq 0$ for all r and this is called ***Doolittle factorisation****.*

Also, A can be expressed as a product of lower-triangular matrix L and unit upper triangular matrix U:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix},$$

and this is called ***Crout factorisation****.*

To solve the linear system $AX = B$ using LU factorisation, we do the following two steps:

1. Using forward substitution to solve the the lower-triangular linear system $LY = B$ for Y.
2. Using backward substitution to solve the upper-triangular linear system $UX = Y$ for X.

Direct LU Factorisation Using Gaussian Elimination Method

The matrix A can be factored directly using Gauss elimination method without any row interchanges. In this case the matrix A is expressed in terms of the identity matrix I follows $A = IA$. We perform the row operations on the matrix A on the right and the resulting matrix it will be the upper triangular matrix U. The multipliers are stored in their appropriate places in the identity matrix on the left which will be the lower triangular matrix L. All this information is summarised in the next theorem.

Theorem 1 (**Direct LU Factorisation Without Row Interchanges**). *Assume that the linear system $AX = B$ can be solved using Gaussian elimination with no row interchanges. Then, the coefficient matrix A can be factored as a product of a lower triangular matrix L and an upper triangular matrix U as follows:*

$$A = LU.$$

The matrix L has 1's on its main diagonal and the matrix has nonzero entries on its main diagonal. After constructing the matrices L and U then the linear system can be solved in the following two steps:

(1). Solve the lower triangular system $LY = B$ for Y using the forward substitution method.

(2). Solve the upper triangular system $UX = Y$ for X using the backward substitution method.

Proof. For proof, see any standard text on numerical analysis or numerical linear algebra. □

The following example explains this type of LU factorisation.

Example 13. *Find the LU factorisation of the following matrix using Gaussian elimination without row interchanges*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix}.$$

Solution. *Writing the matrix A in terms of the identity matrix as follows*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = IA$$

The first row is used to eliminate the elements under the main diagonal (subdiagonal elements) in the first column. The multipliers of the first row are $m_{21} = a_{21}/a_{11} = -2/2 = -1$ *and* $m_{31} = a_{31}/a_{11} = 1/2 = 0.5$*, respectively.*

$$\begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 3 & 6.5 \end{bmatrix}$$

Now, the second row is used to eliminate the entries below the main diagonal in the second column and the multiple of the second row is $m_{32} = a_{32}/a_{22} = 3/7$*. Hence, we have the following LU factorisation of A*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3/7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 6.5 \end{bmatrix} = LU.$$

The LU Factorisation Without Using Gaussian Elimination Method

Example 14. *Solve the following linear system using LU (Doolittle) decomposition*

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 &= -1 \\ -x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Solution: Express the system in matrix form

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 \end{array}\right].$$

Factor A as follows:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Find the values of the entries of matrices L and U. From the first column we have

$$2 = 1u_{11} \implies u_{11} = 2,$$

and

$$1 = l_{21}u_{11} = l_{21}2 \implies l_{21} = 0.5,$$

finally

$$-1 = l_{31}u_{11} = l_{31}2 \implies l_{31} = -0.5.$$

In the second column, we have

$$-3 = 1u_{12} \implies u_{12} = -3,$$

and

$$1 = l_{21}u_{12} + 1u_{22} = -1.5 + u_{22} \implies u_{22} = 2.5,$$

so

$$1 = l_{31}u_{12} + l_{32}u_{22} = (-0.5)(-3) + l_{32}(2.5) \implies l_{32} = -0.2.$$

Finally, in the third column we have

$$1 = 1u_{13} \implies u_{13} = 1,$$

and

$$-1 = l_{21}u_{13} + 1u_{23} = 0.5 + u_{23} \implies u_{23} = -1.5,$$

finally,

$$-1 = l_{31}u_{13} + l_{32}u_{23} + 1u_{33} = -0.5(1) + (-0.2)(-1.5) + u_{33} \implies u_{33} = -0.8.$$

Now, we have the LU factorisation

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & -0.8 \end{bmatrix} = LU.$$

Now, we have the following lower-triangular linear system $LY = B$ for Y

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Write the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0.5 & 1 & 0 & -1 \\ -0.5 & -0.2 & 1 & 0 \end{array}\right].$$

Solve this system by forward substitution to have

$$y_1 = 2, \quad y_2 = -1 - 0.5(y_1) = -1 - 0.5(2) = -2,$$

and

$$y_3 = 0 + 0.5(y_1) + 0.2(y_2) = 0.5(2) + 0.2(-2) = 0.6.$$

Now, we have the following upper-triangular linear system $UX = Y$

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0.6 \end{bmatrix}.$$

Express the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 0 & 2.5 & -1.5 & -2 \\ 0 & 0 & -0.8 & 0.6 \end{array}\right].$$

Finally, use the values of Y to solve the upper-triangular linear system $UX = Y$ by back substitution to have

$$x_3 = \frac{0.6}{-0.8} = \frac{-3}{4}, \quad x_2 = \frac{-2 + 1.5(x_3)}{2.5} = \frac{-2 + 1.5(-3/4)}{2.5} = -5/4,$$

and

$$x_1 = \frac{2 + 3(x_2) - 1(x_3)}{2} = \frac{2 + 3(-5/4) - (-3/4)}{2} = -1/2.$$

Definition 16 (Cholesky Factorisation). *Let A be a real, symmetric and positive definite matrix. Then, it can be* ***factored*** *or* ***decomposed*** *in a unique way $A = LL^T$, in which L is a lower-triangular matrix with a positive diagonal, and is termed* ***Cholesky factorisation****.*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

Example 15. *(a) Determine the Cholesky decomposition of the matrix*

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

(b) Then, use the decomposition from part (a) to solve the linear system

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= -1 \\ -x_1 + 3x_2 + 3x_3 &= -4 \\ 2x_1 + 3x_2 + 2x_3 &= 2 \end{aligned}$$

Solution: Factor A as a product LL^T as follows:

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

From the first column we obtain

$$2 = l_{11}^2 \implies l_{11} = \sqrt{2},$$

$$-1 = l_{21}l_{11} + l_{22}(0) \implies -1 = l_{21}\sqrt{2} \implies l_{21} = \frac{-1}{\sqrt{2}},$$

$$2 = l_{31}l_{11} + l_{32}(0) + l_{33}(0) \implies 2 = l_{31}\sqrt{2} \implies l_{31} = \frac{1}{\sqrt{2}}.$$

Now, from the second column we have

$$3 = l_{21}^2 + l_{22}^2 \implies 3 = \frac{1}{2} + l_{22}^2 \implies l_{22} = \sqrt{\frac{5}{2}},$$

$$3 = l_{31}l_{21} + l_{32}l_{22} + l_{33}(0) \implies 3 = \frac{-1}{2} + l_{32}\sqrt{\frac{5}{2}} \implies l_{32} = \frac{7}{\sqrt{10}}.$$

Finally, from the third column we get

$$2 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies 2 = \frac{1}{2} + \frac{49}{10} + l_{33}^2 \implies l_{33} = \sqrt{\frac{17}{5}}.$$

0.5 Iterative Methods

Direct methods are more efficient in solving linear systems of small dimensions in less computational cost than iterative methods. For large linear systems in particular for sparse linear systems iterative methods are more efficient for solving linear systems in terms of computational cost and effort compared to direct methods. In this section we will study the most common and basic iterative methods for solving linear algebraic systems which are **Jacobi method** and **Gauss-Siedel method**.

0.5.1 Jacobi Method

The general form of **Jacobi iterative method** for solving the ith equation in the linear system $AX = B$ for unknown $x_i, i = 1, , \cdots, n$ is:

$$x_i^k = \sum_{j=1}^{n} \Big(-\frac{a_{ij}x_j^{k-1}}{a_{ii}} \Big) + \frac{b_i}{a_{ii}},\ j \neq i,\ a_{ii} \neq 0,\ \text{for } i = 1, \cdots, n,\ k = 1, \cdots, n.$$

It is also known as **Jacobi iterative process** or **Jacobi iterative technique**

Example 16. *Solve the following linear system using Jacobi iterative method*

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0.5 \\ x_1 + x_2 + 2.5x_3 &= 0 \end{aligned}$$

Solution: These equations can be written in the form

$$\begin{aligned} x_1 &= \frac{-x_2 - x_3}{2}, \\ x_2 &= \frac{0.5 - x_1 - x_3}{3}, \\ x_3 &= \frac{-x_1 - x_2}{2.5}. \end{aligned}$$

Writing these equations in iterative form

$$\begin{aligned} x_1^{k+1} &= \frac{-x_2^k - x_3^k}{2}, \\ x_2^{k+1} &= \frac{0.5 - x_1^k - x_3^k}{3}, \\ x_3^{k+1} &= \frac{-x_1^k - x_2^k}{2.5}. \end{aligned}$$

Let us start with initial guess $P_0 = (x_1^0, x_2^0, x_3^0) = (0, 0.1, -0.1)$. Substituting these values in the right-hand side of each equation in above to find the new iterations

$$\begin{aligned} x_1^1 &= \frac{-x_2^0 - x_3^0}{2} = \frac{-0.1 - (-0.1)}{2} = \frac{-0.1 + 0.1}{2} = 0, \\ x_2^1 &= \frac{0.5 - x_1^0 - x_3^0}{3} = \frac{0.5 - 0 - (-0.1)}{3} = 0.2, \\ x_3^1 &= \frac{-x_1^0 - x_2^0}{2.5} = \frac{-0 - 0.1}{2.5} = -0.04. \end{aligned}$$

Now, the new point $P_1 = (x_1^1, x_2^1, x_3^1) = (0, 0.2, -0.04)$ is used in the Jacobi iterative form to find the next approximation P_2

$$
\begin{aligned}
x_1^2 &= \frac{-x_2^1 - x_3^1}{2} = \frac{-0.2 + 0.04}{2} = \frac{-0.16}{2} = -0.08,\\
x_2^2 &= \frac{0.5 - x_1^1 - x_3^1}{3} = \frac{0.5 + 0.04}{3} = \frac{0.54}{3} = 0.18,\\
x_3^2 &= \frac{-x_1^1 - x_2^1}{2.5} = \frac{-0 - 0.2}{2.5} = \frac{-0.2}{2.5} = -0.08.
\end{aligned}
$$

The new point $P_2 = (x_1^2, x_2^2, x_3^2) = (-0.08, 0.18, -0.08)$ is closer to the solution than P_0 and P_1 and is used to find P_3

$$
\begin{aligned}
x_1^3 &= \frac{-x_2^2 - x_3^2}{2} = \frac{-0.18 + 0.08}{2} = \frac{-0.1}{2} = -0.05,\\
x_2^3 &= \frac{0.5 - x_1^2 - x_3^2}{3} = \frac{0.5 + 0.08 + 0.08}{3} = \frac{0.66}{3} = 0.22,\\
x_3^3 &= \frac{-x_1^2 - x_2^2}{2.5} = \frac{0.08 - 0.18}{2.5} = \frac{-0.1}{2.5} = -0.04.
\end{aligned}
$$

This Jacobi iteration process generates a sequence of points $\{P_n\} = \{(x_1^n, x_2^n, x_3^n)\}$ that converges to the solution $(x_1, x_2, x_3) = (-3/38, 4/19, -1/19) = (-0.078947368421053, 0.210526315789474, -0.052631578947368)$. The outline of the results is given in the Table 1.

n	x_1^n	x_2^n	x_3^n
0	0.000000000000000	0.100000000000000	−0.100000000000000
1	0.000000000000000	0.200000000000000	−0.040000000000000
2	−0.080000000000000	0.180000000000000	−0.080000000000000
3	−0.050000000000000	0.220000000000000	−0.040000000000000
4	−0.090000000000000	0.196666666666667	−0.068000000000000
5	−0.064333333333333	0.219333333333333	−0.042666666666667
6	−0.088333333333333	0.202333333333333	−0.062000000000000
7	−0.070166666666667	0.216777777777778	−0.045600000000000
8	−0.085588888888889	0.205255555555556	−0.058644444444444
9	−0.073305555555556	0.214744444444444	−0.047866666666667
10	−0.083438888888889	0.207057407407407	−0.056575555555556
11	−0.075240925925926	0.213338148148148	−0.049447407407407
12	−0.081945370370370	0.208229444444444	−0.055238888888889
13	−0.076495277777778	0.212394753086420	−0.050513629629630
14	−0.080940561728395	0.209002969135802	−0.054359790123457
15	−0.077321589506173	0.211766783950617	−0.051224962962963
16	−0.080270910493827	0.209515517489712	−0.053778077777778
17	−0.077868719855967	0.211349662757202	−0.051697842798354
18	−0.079825909979424	0.209855520884774	−0.053392377160494
19	−0.078231571862140	0.211072762379973	−0.052011844362140
20	−0.079530459008916	0.210081138741427	−0.053136476207133

Table 1: Jacobi Iterative Solution of Example 16

0.5.2 Gauss-Siedel Method

An improvement of Jacobi method can be made by using the recent values x_i^k, $i, k = 1, \cdots, n$, in the calculations once their values are obtained. This improvement is called **Gauss-Siedel iterative method** and its general form for solving the *ith* equation in the linear system $AX = B$ for unknown $x_i, i = 1, , \cdots, n$ is:

$$x_i^k = \sum_{j=1}^{i-1} \Big(-\frac{a_{ij} x_j^k}{a_{ii}} \Big) + \sum_{j=i+1}^{n} \Big(-\frac{a_{ij} x_j^{k-1}}{a_{ii}} \Big) + \frac{b_i}{a_{ii}}, \ j \neq i, \ a_{ii} \neq 0,$$

for $i = 1, \cdots, n$, and $k = 1, \cdots, n$.

It is also known as **Gauss-Siedel iterative process** or **Gauss-Siedel iterative technique**

Example 17. *Solve the following linear system using Gauss-Siedel iterative method*

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -1 \\ x_1 + x_2 + 6x_3 &= 1 \\ 3x_1 + 3x_2 + 5x_3 &= 4 \end{aligned}$$

.

Solution: Rearrange the system in above such that the coefficient matrix is strictly diagonally dominant

$$\begin{aligned} 3x_1 + 3x_2 + 5x_3 &= 4 \\ 2x_1 - 4x_2 + x_3 &= -1 \\ x_1 + x_2 + 6x_3 &= 1 \end{aligned}$$

.

These equations can be written in the form

$$\begin{aligned} x_1 &= \frac{4 - 3x_2 - 5x_3}{3}, \\ x_2 &= \frac{-1 - 2x_1 - x_3}{-4} = \frac{1 + 2x_1 + x_3}{4}, \\ x_3 &= \frac{1 - x_1 - x_2}{6}. \end{aligned}$$

This suggests the following Gauss-Siedel iterative process

$$\begin{aligned} x_1^{n+1} &= \frac{4 - 3x_2^n - 5x_3^n}{3}, \\ x_2^{n+1} &= \frac{1 + 2x_1^{n+1} + x_3^n}{4}, \\ x_3^{n+1} &= \frac{1 - x_1^{n+1} - x_2^{n+1}}{6}. \end{aligned}$$

We start with initial guess $P_0 = (x_1^0, x_2^0, x_3^0) = (1, 0.1, -1)$. Substitute $x_2^0 = 0.1$ and $x_3^0 = -1$ in the first equation and have

$$x_1^1 = \frac{4 - 3x_2^0 - 5x_3^0}{3} = \frac{4 - 3(0.1) - 5(-1)}{3} = \frac{8.7}{3} = 2.9.$$

Then, substitute the new value $x_1^1 = 2.9$ and $x_3^0 = -1$ into the second equation to obtain

$$x_2^1 = \frac{1 + 2x_1^1 + x_3^0}{4} = \frac{1 + 2(2.9) + (-1)}{4} = 1.45.$$

Finally, substitute the new values $x_1^1 = 2.9$ and $x_2^1 = 1.45$ in the third equation and get

$$x_3^1 = \frac{1 - x_1^1 - x_2^1}{6} = \frac{1 - 2.9 - 1.45}{6} = \frac{-3.35}{6} = -0.558333333333333.$$

Now, we have the now point $P_1 = (x_1^1, x_2^1, x_3^1) = (2.9, 1.45, -0.558333333333333)$ is used to find the next approximation P_2.

Substitute $x_2^1 = 1.45$ and $x_3^1 = -0.558333333333333$ in the first equation and get

$$\begin{aligned} x_1^2 &= \frac{4 - 3x_2^1 - 5x_3^1}{3} = \frac{4 - 3(1.45) - 5(-0.558333333333333)}{3} \\ &= \frac{2.441666666666666}{3} = 0.813888888888889. \end{aligned}$$

Then, substitute the new value $x_2^1 = 0.813888888888889$ and $x_3^1 = -0.558333333333333$ into the second equation to obtain

$$\begin{aligned} x_2^2 &= \frac{1 + 2x_1^2 + x_3^1}{4} = \frac{1 + 2(0.813888888888889) + (-0.558333333333333)}{4} \\ &= \frac{2.069444444444445}{4} = 0.517361111111111. \end{aligned}$$

Finally, substitute the new values $x_2^1 = 0.813888888888889$ and $x_2^2 = 0.517361111111111$ in the third equation and get

$$\begin{aligned} x_3^2 &= \frac{1 - x_1^2 - x_2^2}{6} = \frac{1 - 0.813888888888889 - 0.517361111111111}{6} \\ &= \frac{-0.331250000000000}{6} = -0.055208333333333. \end{aligned}$$

This iteration process generates a sequence of points $\{P_n\} = \{(x_1^n, x_2^n, x_3^n)\}$ that converges to the solution $(x_1, x_2, x_3) = (32/39, 25/39, -1/13) = (0.820512820512820, 0.641025641025641, -0.076923076923077)$. The results are given in the Table 2.

n	x_1^n	x_2^n	x_3^n
0	1.000000000000000	0.100000000000000	−1.000000000000000
1	2.900000000000000	1.450000000000000	−0.558333333333333
2	0.813888888888889	0.517361111111111	−0.055208333333333
3	0.907986111111111	0.690190972222222	−0.099696180555556
4	0.809302662037037	0.629727285879630	−0.073171657986111
5	0.825558810763889	0.644486490885417	−0.078340883608218
6	0.819414981794946	0.640122269995419	−0.076589541965061
7	0.820860299946349	0.641282764481909	−0.077023844071376
8	0.820423642303718	0.640955860134015	−0.076896583739622
9	0.820538446098689	0.641045077114439	−0.076930587202188
10	0.820505901555874	0.641020303977390	−0.076921034255544
11	0.820514753115183	0.641027117993706	−0.076923645184815
12	0.820512290647653	0.641025234027623	−0.076922920779213
13	0.820512967271065	0.641025753440729	−0.076923120118632
14	0.820512780090325	0.641025610015504	−0.076923065017638
15	0.820512831680559	0.641025649585870	−0.076923080211072
16	0.820512817432583	0.641025638663523	−0.076923076016018
17	0.820512821363173	0.641025641677582	−0.076923077173459
18	0.820512820278183	0.641025640845727	−0.076923076853985
19	0.820512820577581	0.641025641075294	−0.076923076942146
20	0.820512820494949	0.641025641011938	−0.076923076917814

Table 2: Gauss-Siedel Iterative Solution of Example 17

Exercises

Exercise 2. *Solve Example 9 Using Gauss elimination with forward substitution method. Compare the solution with solution of the same example.*

Exercise 3. *Solve Example 10 Using Gauss elimination with backward substitution method. Compare the solution with solution of the same example.*

Exercise 4. *Repeat Example 16 with Gauss-Siedel iteration. Compute five iterations and compare them with Jacobi iterations in the same example.*

Exercise 5. *Redo Example 17 with Jacobi iteration. Compute five iterations and compare them with Gauss-Siedel iterations in the same example.*

Exercise 6. *Use Gauss elimination with backward substitution method and three-digit rounding arithmetic to solve the following linear system*

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - 3x_3 &= -2 \\ x_1 + 5x_2 + 3x_3 &= 10 \end{aligned}$$

.

Exercise 7. *(a) Determine the LU factorisation for matrix A in the linear system $AX = B$, where*

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & -1 & 1 \\ -4 & 1 & -2 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

(b) Then use the factorisation to solve the system

$$\begin{aligned} -x_1 + x_2 - 2x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \\ -4x_1 + x_2 - 2x_3 &= 4 \end{aligned}$$

.

Exercise 8. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} -4x_1 - x_2 - 2x_3 &= -9 \\ -x_1 - x_2 + 3x_3 &= 9 \\ -2x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

.