جامعة تكريت

كلية التربية للبنات

قسم الرياضيات

تفاضل وتكامل

التكامل المحدد وتطبيقاته

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CHAPTER 10 : The Definite Integrals and Its Application

10.1 Definite Integrals:

If $a, b \in \mathbb{R}$ and $F(x)$ is an anti-derivative for $f(x)$, then:

$$\int_{a}^{b} f(x) = F(x)\Big]_{a}^{b} = F(b) - F(a)$$

a is called lower limit, and b is called upper limit for the integral.

Properties for Definite Integrals:

1. $\int_{a}^{b} (k_1 f \mp k_2 g) = k_1 \int_{a}^{b} f \mp k_2 \int_{a}^{b} g$
2. $\int_{a}^{b} f = \int_{a}^{c} f + \int_{c}^{b} f$, where $c \in [a, b]$
3. $\int_{a}^{b} f = -\int_{b}^{a} f$
4. $\int_{a}^{a} f = 0$
5. If f is an <u>EVEN</u> function $\longrightarrow \int_{a}^{a} f = 2\int_{0}^{a} f$
6. It f is an <u>ODD</u> function $\Longrightarrow \int_{a}^{a} {}_{a}f = 0$

Examples: Evaluate the following integrals:

$$1)\ \int_1^6 (3x^2 + 2x)dx$$
$$= \left(3\frac{x^3}{3} + 2\frac{x^2}{2}\right)\Bigg]_1^6$$
$$= (x^3 + x^2)]_1^6$$
$$= (6^3 + 6^2) - (1^3 + 1^2) = 252 - 2 = 250$$

$$2)\ \int_0^2 \sqrt{4x+1}dx$$
$$= \frac{1}{4}\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^2$$
$$= \frac{1}{6}\sqrt{(4x+1)^3}\Bigg]_0^2$$
$$= \frac{1}{6}\left(\sqrt{(4*2+1)^3} - \sqrt{(4*0+1)^3}\right) = \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$$

3)
$$\int_0^{\pi} \sin(x)dx$$
$$= -\cos(x)]_0^{\pi}$$
$$= -\cos(\pi) - \cos(0) = -(-1) + 1 = 2$$

4)
$$\int_0^{\frac{\pi}{4}} \csc^2(x)dx$$
$$= \cot(x)]_0^{\frac{\pi}{4}} = \left(\cot\left(\frac{\pi}{4}\right) - \cot(0)\right)$$
$$= 1 - \infty = -\infty$$

5)
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(2)}{\sin^3\ x)}dx$$
$$= \int_{\frac{4}{4}}^{\frac{3}{4}}(\cos(x)\sin^{-3}(x)dx$$
$$= \frac{\sin^2(x)}{4}\Bigg|_{\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\left[\frac{1}{\sin^2(x)}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\left[\frac{1}{\sin^2\left(\frac{\pi}{2}\right)} - \frac{1}{\sin^2\left(\frac{\pi}{4}\right)}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{1} - \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\right] = \frac{-1}{2}[1-2] = \frac{1}{2}$$

Problems (10.1): Evaluate the following integrals:

1. $\int_{-1}^{1}(x+1)^2dx$
2. $\int_{1}^{2}\frac{2t}{t^2+1}dt$
3. $\int_{0}^{1}(x^2-2x+3)dx$
4. $\int_{-1}^{3}(x^2-1)dx$
5. $\int_{1}^{2}(2w+5)dw$
6. $\int_{0}^{\frac{\pi}{4}}\sec^2(x)dx$
7. $\int_{1}^{2}\left(3-\frac{6}{x^2}\right)dx$
8. $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}}(1+\tan^2(\theta))d\theta$
9. $\int_{0}^{1}\sqrt{1+x}dx$
10. $\int_{1}^{3}5e^{2z}dz$
11. $\int_{1}^{2}\frac{1}{f}dt$
12. $\int_{0}^{\pi}\cos\left(\frac{w}{2}\right)dw$

13. $\int_0^{\pi} \sin^2(\theta) d\theta$

14. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cot \theta d\theta$

15. $\int_0^{\frac{2x}{W}} \cos^2(wt) dt$

16. $\int_0^{\frac{\pi}{2}} \csc(\theta) \cot(\theta) d\theta$

17. $\int_0^1 \frac{1}{(2x+1)^3} dx$

18. $\int_0^{\frac{\pi}{3}} x \sin(x) dx$

19. $\int_0^{\frac{\pi}{6}} \frac{\sin(2z)}{\cos^2(2z)} dz$

20. $\int_0^{\frac{\pi}{2}} \theta \cos(3\theta) d\theta$

21. $\int_0^5 2^x dx$

22. $\int_0^3 x\sqrt{4x+2} dx$

23. $\int_{-\pi}^0 \cos^2(2\theta) d\theta$

24. $\int_0^{\frac{1}{2}} x \tan^{-1} x dx$

25. $\int_0^4 \pi e^{3x} dx$

26. $\int_0^3 x\sqrt{4x+2} dx$

10.2 Multiple Integrals:

The multiple integral is a definite integral of a function of more than one real variable, for example, $f(x, y)$ or $f(x, y, z)$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called double integrals, and integrals of a function of three variables over a region of $\mathbb{R}^3$ are called triple integrals. i.e.,

$$\int_y \int_x f(x, y)dxdy$$

$$\int_z \int_y \int_x f(x, y, z)dxdydz$$

Examples: Evaluate the following integrals:

$$1) \int_0^{\pi} \int_0^{x} \sin(y)dydx$$

$$= \int_0^{\pi} -\cos(y)\Big]_0^{x} dx$$

$$= \int_0^{\pi} (-\cos(x) + 1)dx$$

$$= (-\sin(x) + x)]_0^{\pi}$$

$$= (-\sin(\pi) + \pi) - (-\sin(0) + 0)$$

$$= (-0 + \pi) - (0 + 0) = \pi$$

2. $\int_0^1 \int_{\frac{1}{2}}^0 \int_{2-y}^{x+y+1} dzdyd.x$

$$= \int_0^1 \int_{\frac{-1}{2}}^0 z \Big]_{x-y}^{x+y+1} dydx$$

$$= \int_0^1 \int_{\frac{-1}{2}}^0 ((x + y + 1) - (x - y))dydx$$

$$= \int_0^1 \int_{\frac{-1}{2}}^0 (1 + 2y)dydx$$

$$= \int_0^1 (y + y^2)\Big]_{\frac{-1}{2}}^0 dx$$

$$= \int_0^1 \left[0 - \left(\frac{-1}{2} + \left(\frac{-1}{2}\right)^2\right)\right] dx$$

$$= \int_0^1 \frac{1}{4} dx$$

$$= \frac{1}{4} x \Big]_0^1$$

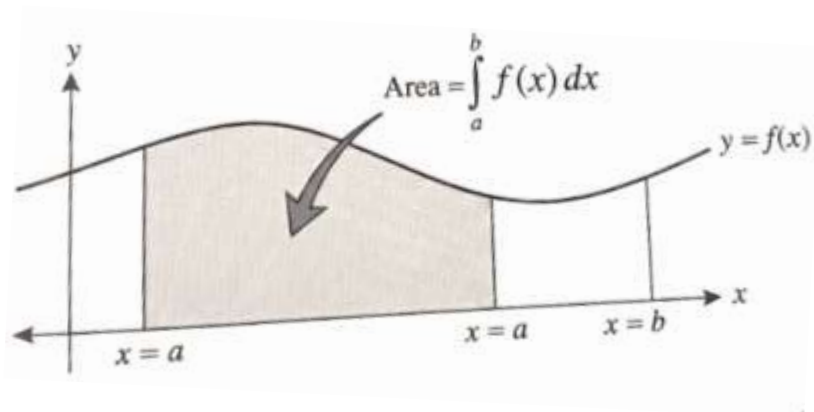
$$= \frac{1}{4}(1 - 0) = \frac{1}{4}$$

<u>Problems (10.2)</u>: Evaluate the following integrals:

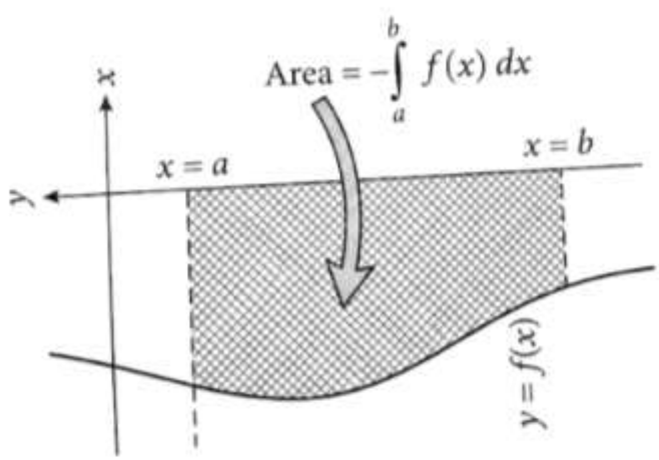
1. $\int_0^{\pi} \int_{-1}^{0} \frac{2}{\pi} dydx$
2. $\int_{-1}^{1} \int_0^{2} (1 - 6x^2 y)dxdy$
3. $\int_0^{2} \int_{-1}^{1} \int_y^{x+2} dzdydx$
4. $\int_0^{2} \int_0^{4-y^2} \int_0^{y} \left(\frac{3}{2}\right) dzdxdy$
5. $\int_{-1}^{1}, \int_0^{1-x} \int_{4r^2}^{2} \pi dzdydx$
6. $\int_0^{2} \int_{-1}^{1} \int_y^{x+2} dzdydx$

10.3 Area Under a Curve

The area under a curve between two points can be found by doing a definite integral between the two points. To find the area under the curve $y = f(x)$ between $x = a$ and $x = b$, integrate $y = f(x)$ between the limits of a and b.



Remark: If the area is above x-axis, then the area is positive, and if the area under the x-axis, the area is negative, so we should change the sign to positive value by adding a negative sign or by taking the absolute value.



<u>Remark</u>: To avoid the negative value, we will take the absolute value:

$$\text{Area } = \left| \int_{x=a}^{x=b} f(x)dx \right|$$

Example (1): Find the area bounded by $y = x^2$ and $x = 1$ and $x = 3$?

<u>Solution:</u>

$$\text{Area } = \left| \int_{x=1}^{x=3} x^2 dx \right|$$

$$= \left| \left. \frac{x^3}{3} \right]_{x=1}^{x=3} \right|$$

$$= \left| \frac{3^3}{3} - \frac{1^3}{3} \right|$$

$$= \left| 9 - \frac{1}{3} \right| = 8\frac{2}{3} \text{ unit }^2$$

<u>Example</u> (2): Find the total area between the curve $y = x^3$ and $x = -2$ and $x = 2$?

<u>Solution</u>:

If we simply integrated x^3 between $x = -2$ and $x = 2$, we would get:

$$\text{Area } = \left| \int_{x=2}^{x=2} x^3 \, dx \right| = \left| \frac{x^4}{4} \right|_{x=-2}^{x=2} = \left| \frac{16}{4} - \frac{16}{4} \right| = 0$$

So, instead we have to split the graph up and do two separate integrals:

$$A1 = \left| \int_{x=0}^{x=2} x^3 dx \right| = \left| \frac{x^4}{4} \right]_0^2 \Bigg| = \left| \frac{16}{4} - 0 \right| = 4$$

$$A2 = \left| \int_{x=-2}^{x=0} x^3 dx \right| = \left| \frac{x^4}{4} \right]_{-2}^0 \Bigg| = \left| 0 - \frac{16}{4} \right| = |-4| = 4$$

Hence, Area $= A1 + A2 = 4 + 4 = 8u_{nit^2}$

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<u>Example (3):</u> Find the area bounded by the line $x + y = 1$ and the coordinate axes?

<u>Solution</u>:

$$\because x + y = 2 \Longrightarrow y = 2 - x$$

$$y = 0 \longrightarrow x = 2 \Longrightarrow (2,0)$$

$$\text{Area } = \left| \int_{x=0}^{x=2} (2 - x) dx \right|$$

$$= \left| \left(x - \frac{x^2}{2} \right) \right]_{x=0}^{x=2} |$$

$$= \left| (0 - 0) - \left(2 - \frac{2^2}{2} \right) \right|$$

$$= |-2 + 2| = [4 \text{ unit }^2$$

Another way:

$$\because x+y=2 \Longrightarrow x=2-y$$
$$x=0 \longrightarrow y=2 \Longrightarrow (2,0)$$
$$\text{Area } = \left|\int_{y=0}^{y=2} (2-y)dy\right|$$
$$= \left| y-\frac{y^2}{2}\right]_{y=0}^{y=2} |$$
$$= \left|(0-0)-\left(2-\frac{2^2}{2}\right)\right|$$
$$= |-2+2| = 4 \text{ unit }^2$$

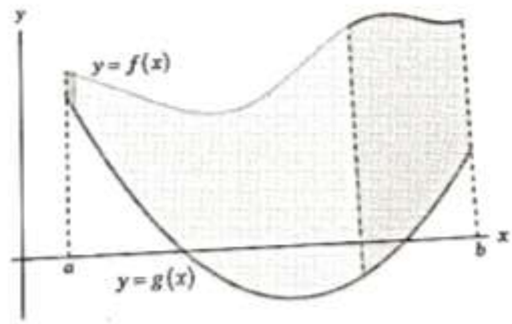
Problems (10.3):

1. Find the total area bounded by the curve $y = x^3 - 4x$ and x-axis.
2. Find the area bounded by $y = y4 - x^2$ and x -axis.
3. Find the area bounded by $x = y^2 - y^3$ and y -axis.
4. Find the area bounded by $\sqrt{x} + \sqrt{y} = 1$ and the two axes.
5. Prove that the area under one carve of $y = \sin(x)$ equals to 2 units ^{2}.
6. Find the area bounded by $y = x^2 - 4x$ and x -axis.
7. Find the area bounded by $x = 8 - 2y - y^2$ and y-axis.

10.4 Area Between Two Curves

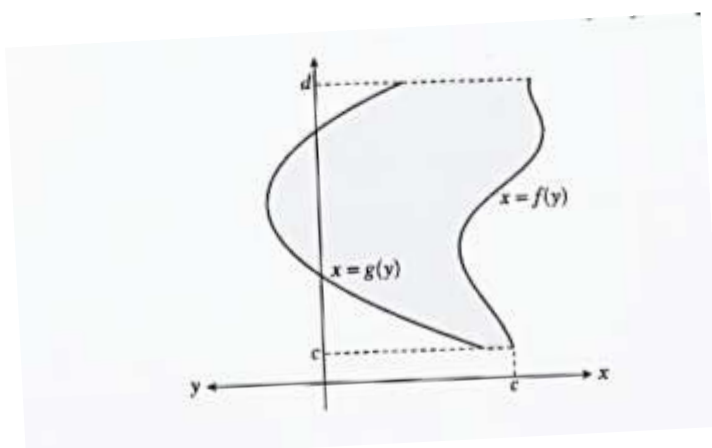
In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



$$\text{Area} = \left| \int_{x=a}^{x=b} [f(x) - g(x)]dx \right|$$

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



$$\text{Area} = \left| \int_{y-a}^{y=b} [f(y) - g(y)]dy \right|$$

Example (1): Find the area between the curve $y = 2 - x^2$ and the line $y = -x$?

Solution:

$$y_1 = y_2 \Longrightarrow 2 - x^2 = -x$$
$$\Longrightarrow x^2 - x - 2 = 0$$
$$\Longrightarrow (x-2)(x+1) = 0 \Longrightarrow x = 2 \text{ and } x = 1$$
$$\text{Area } = \left|\int_{x=-1}^{x=2} [f(x) - g(x)]dx\right|$$
$$= \left|\int_{x=-1}^{x=2} [(2 - x^2) - x]dx\right|$$
$$= \left|\int_{x=-1}^{x=2} [2 - x^2 - x]dx\right|$$
$$= \left|\left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{x=2}^{x=2}\right| = \left[\left[\frac{27}{6}\right]\right] \text{ units }^2$$

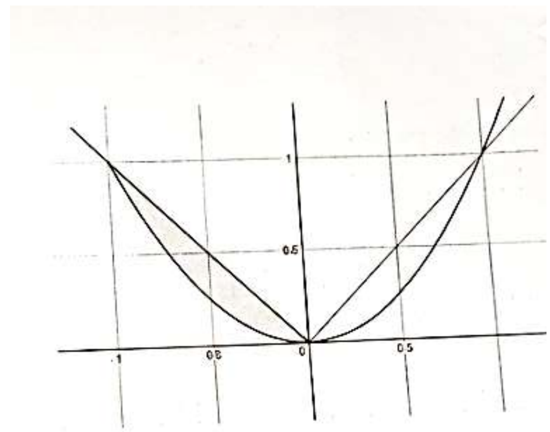
Example (2): Find the area of the triangular shaped region in the first quarter bounded by the y-axis and the curves $y = \sin(x)$ and $y = \cos(x)$?

Solution:

$$\text{Area } = \left|\int_{x=0}^{x=\frac{\pi}{4}} [f(x) - g(x)]dx\right|$$
$$= \left|\int_{x=0}^{x=\frac{\pi}{4}} [\cos(x) - \sin(x)]dx\right|$$
$$= \left|[\sin(x) + \cos(x)]_{x=0}^{x=\frac{\pi}{4}}\right|$$
$$= \left|\left[\left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right) - (\sin(0) + \cos(0))\right]\right|$$
$$= \left|\left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)\right]\right|$$
$$= \frac{2}{\sqrt{2}} + 1 \text{ units }^2$$

Example (3): Find the area bounded helwent the two carves y=x^2 and $y = |x|$?

Solution:



$y_1 = y_2$

$\rightarrow |x| = x^2 \rightarrow \sqrt{x} = x^2$

$\longrightarrow x^2 = x^4 \longrightarrow x^2 - x^4 = 0$

$\longrightarrow x^2(1 - x^2) = 0$

$\longrightarrow x^2(1 - x)(1 + x) = 0$

$x^2 = 0 \Longrightarrow x = 0 \longrightarrow (0,0)$

$(x - 1) = 0 \Rightarrow x = 1 \longrightarrow (1,1)$

$(x + 1) = 0 \Longrightarrow x = -1 \longrightarrow (-1,1)$

$$\text{A1} = \left|\int_{x=-1}^{x=0} [f(x) - g(x)]dx\right|$$

$$= \left|\int_{x=-1}^{x=0} [-x - x^2]dx\right|$$

$$= \left|\left[-\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=-1}^{x=0}\right| = \left|\left[0 - \left(\frac{-1}{2} - \frac{-1}{3}\right)\right]\right| = \left|\frac{1}{2} - \frac{1}{3}\right| = \frac{1}{6}$$

$$\text{A2} = \left|\int_{x=0}^{x=1} [f(x) - g(x)]dx\right|$$

$$= \left|\int_{x=0}^{x=1} [x - x^2]dx\right|$$

$$= \left|\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=1}^{x=1}\right| = \left|\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right]\right| = \left|\frac{1}{2} - \frac{1}{3}\right| = \frac{1}{6}$$

$$\text{Arca} = \text{A1} + \text{A2} = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \left[\frac{1}{3}\right] \text{ units}^2$$

Example (4): Find the area bounded between carve $y = \frac{1}{x^2}$ and the two lines $y = 1$ and $y = 3$?

Solution:

$$y = \frac{1}{x^2}$$
$$\longrightarrow x^2 = \frac{1}{y}$$
$$\longrightarrow x = \mp \frac{1}{\sqrt{y}}$$

$$\text{Area } = \left| \int_{y=1}^{y=3} [f(x) - g(x)]dx \right|$$
$$=$$

<u>Problems (10.4):</u>

1. Find the area bounded by the curve $y = \sqrt{x}$ and the line $y = x$.
2. Find the area bounded by the curve $y = x^3$ and the lines $x = -1$ and $x = -3$.
3. Find the area bounded by the curves $y = x - x^2$ and $y = x^2 - x$.
4. Find the area bounded by the curve $x = 4y - y^2 - 3$ and the line $x = -3$.
5. Find the area bounded by the curve $y = x^3$ and the line $y = x$ in the first quarter.
6. Find the area bounded between carve $y = \frac{1}{x}$ and the two lines $y = 2$ and $y = 3$?
7. Find the area bounded by the curves $y = e^x$ and $y = e^{-x}$ and the lines $y = 2$ and $y = 4$.
8. Find the area bounded by the curves $y = x^2 + 2$ and the $y = x + 5$.