جامعة تكريت

كلية التربية البنات

قسم الرياضيات

تفاضل وتكامل

التكامل بتجزئة الكسور

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٩. ٣ Integration by Partial Fractions:

If the integrand (a function that is to be integrated/the expression after the integral sign) is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in partial fractions before integration takes place.

Remark(1):

IF the "degree of denominator " > " degree of enumerator", then we use the partial fraction as follows:

i) Ife have $\frac{1}{ax^2+bx+c}$ and we are unable to analyze it, then its numerator will be always one degree less.

Examples:

- $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{1}{x+2}$
- $\frac{1}{(x-3)(x^2+3x+2)} = \frac{1}{(x-3)(x+1)(x+2)} = \frac{1}{x-3} + \frac{1}{x+1} + \frac{C}{x+2}$
- $\frac{1}{(x-3)(x^2+x+1)} = \frac{1}{x-3} +$
- $\frac{1}{(x-3)(x^3+x^2+2)} = \frac{1}{x-3} + \frac{1}{x^3+x^2+2}$
- $\frac{1}{(x-3)(x^2+5)(x^4+3)} = \frac{1}{x-3} + \frac{1}{x+5} + \frac{1}{x^4+3}$

ii) If we have $\frac{1}{(x-a)^n}$ where n is a positive constant, then we analyze it as follows:

$$\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \cdots + \frac{A_n}{(x-a)^n}$$

Examples:

- $\frac{1}{(x-3)^2} = \frac{1}{x-3} + \frac{1}{(x-3)^2}$
- $\frac{1}{(x-3)^3} = \frac{1}{x-3} + \frac{1}{(x-3)^2} + \frac{1}{(x-3)^3}$
- $\frac{1}{(x-3)^4} = \frac{1}{(x-3)^2} + \frac{1}{(x-3)^3} + \frac{1}{(x-3)^4}$

Examples:

1) $\int \frac{1}{(x^2-2x-3)}\, dx = ?$

$$\frac{1}{(x^2-2x-3)} = \frac{1}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

$$= \frac{Ax + A + B - 3B}{(x-3)(x+1)} = \frac{(A+B)x + (A-3B)}{(x-3)(x+1)}$$

$$\Longrightarrow A + B = 0$$

$$\mp A \pm 3B = \mp 1$$

_____(by subtracting)

$$\Longrightarrow 4B = -1 \Longrightarrow \boxed{B = \frac{-1}{4}}$$

$$\because A + B + 0 \Longrightarrow A = -B \Longrightarrow \boxed{A = \frac{1}{4}}$$

$$\frac{1}{(x^2-2x-3)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} = \frac{\frac{1}{4}}{(x-3)} + \frac{\frac{-1}{4}}{(x+1)}$$

$$\therefore \int \frac{1}{(x^2-2x-3)}\, dx$$

$$= \int \frac{\frac{1}{4}}{(x-3)} + \frac{\frac{-1}{4}}{(x+1)}\, dx$$

$$= \frac{1}{4}\int \frac{1}{(x-3)}\, dx + \frac{-1}{4}\int \frac{1}{(x+1)} dx$$

$$= \boxed{\frac{1}{4}\, In|x-3| + \frac{-1}{4} In|x+1| + C}$$

2) $\int \frac{1}{x^2(x-2)}\, dx = ?$

$$\frac{1}{x^2(x-2)} = \frac{A}{x} + \frac{C}{x-2}$$

$* x^2(x-2) \qquad 1 = Ax(x-2) + B(x-2) + C^2$

$Let\ x = 0 \implies \boxed{B = \frac{-1}{2}}$

$Let\ x = 2 \implies \boxed{C = \frac{1}{4}}$

Comparing cofficients of x^2 gives

$A + C = 0 \implies \boxed{A = \frac{-1}{4}}$

$$\therefore \int \frac{1}{x^2(x-2)}\, dx = \int \left(\frac{\frac{-1}{4}}{x} + \frac{\frac{-1}{2}}{x^2} + \frac{\frac{1}{4}}{x-2} \right) dx$$

$$= \frac{-1}{4} \int \frac{1}{x}\, dx + \frac{-1}{2} \int \frac{1}{x^2}\, dx + \frac{1}{4} \int \frac{1}{x-2}\, dx$$

$$= \boxed{\frac{-1}{4}\, In|x| + \frac{-1}{2}\, \frac{x^{-1}}{-1} + Inx|x-2| + C}$$

3) $\int \frac{1}{x^2+3x-4} dx = ?$

$$= x^2 + 3x - 4 = (x+4)(x-1)$$

$$\frac{1}{(x-4)(x-1)} = \frac{1}{x+4} + \frac{1}{x-1} = \frac{1}{1+4} = \frac{1}{5}$$

$$\int \left(\frac{\frac{1}{5}}{x+4} + \frac{\frac{1}{5}}{x-1} \right) dx$$

$$= \boxed{\frac{1}{5}\, In\,|x-1| - \frac{1}{5}\, In\,|x+4| + C = \frac{1}{5}\, In \left|\frac{x-1}{x+4}\right| + C}$$

4) $\int \frac{x^2}{(2x+1)(x+2)^2}\,dx = ?$

$$= \frac{A}{(2x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$* (2x+1)(x+2)^2 \quad x^2 = A(x+2)^2 + B(2x+1)(x+2) + C(2x+1)$

$\longrightarrow$

$Let\ x = -2 \implies C$

Let x = $\frac{-1}{2} \implies \boxed{A\frac{1}{9}}$

Comparing cofficients of x^2 gives

$A + 2B = 1 \implies \frac{1}{9} + 2B = 1 \implies \boxed{B = \frac{4}{9}}$

$$\therefore \int \frac{x^2}{(x+2)^2}\,dx$$

$$= \int \left(\frac{\frac{1}{9}}{(2x+1)} + \frac{\frac{4}{9}}{(x+2)} + \frac{\frac{-4}{3}}{(x+2)^2} \right)$$

$$= \frac{1}{9}\int \frac{1}{(2x+1)}\,dx + \frac{4}{9}\int \frac{1}{(x+2)}\,dx + \frac{-4}{3}\int \frac{1}{(x+2)^2}\,dx$$

$$= \frac{1}{18}\int \frac{2}{(2x+1)}\,dx + \frac{4}{9}\int \frac{1}{(x+2)}\,dx + \frac{-4}{3}\int (x+2)^{-2}\,dx$$

$$= \boxed{\frac{1}{18}\,In\,(2x+1) + \frac{4}{9}\,In\,(x+2) + \frac{-4}{3}.\frac{(x+2)^{-1}}{-1} + C}$$

Remark(2):

IF the "degree of enumerator" ≥ "degree of denominator", then we use the long division as follows:

- $\frac{x+4}{x+8} = 1 + \frac{-4}{x^2+8}$
- $\frac{x^2+4}{x^2+8} = 1 + \frac{-4}{x^2+8}$
- $\frac{x^2+4}{x+8} = (x-8) + \frac{68}{x-8}$
- $\frac{x^3+4}{x+8} = (x^2 - 4x + 32) + \frac{252}{x-8}$

Examples:

1) $\int \frac{x-3}{x+3} dx = ?$

$$= \int \frac{(x+5)-8}{x+5} = \frac{x+5}{x+5} - \frac{-8}{x+5} = 1 - \frac{8}{x+5}$$

$$\int (1 -) dx$$

$$\int 1dx \quad \int \frac{8}{x+5} dx = x - 8\, In\, |x+5| + C$$

2) $\int \frac{5-x^2}{16+x^2} dx = ?$

$$\int (-1 + \frac{21}{x^2+16} dx = \int -1dx + 21 = \int \frac{1}{x^2+4} dx$$

$$-x + 21\ \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + C$$

3) $\int \frac{x^2-1}{x^2+9} dx = ?$

$$\frac{x^2-1}{x^2+9} = \frac{x^2+9-9-1}{x^2+9} = \frac{x^2+9-10}{x^2+9} = \frac{10}{x^2+9} = 1 - \frac{10}{x^2+9}$$

$$\int \left(1 - \frac{10}{x^2+9}\right) dx$$

$$\int 1dx - 10 \int \frac{1}{x^2+3^2} dx$$

$$\int \frac{1}{x^2+3^2} dx = \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)$$

$$\int \left(1 - \frac{10}{x^2+9}\right) dx = x - \frac{10}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

4) $\int \frac{x^2+4}{x+8} dx = ?$

$$\because \frac{x^2+4}{x+8} = (x-8) + \frac{68}{x+8}$$

$$\int \frac{x^2+4}{x+8} dx = \int \left((x-8) + \frac{68}{x+8}\right) dx$$

$$= \int \left((x-8) + \frac{68}{x+8}\right) dx$$

$$= \int (x-8)dx + \int \frac{68}{x+8} dx$$

$$= \boxed{\frac{x^2}{2} - 8x + 68In\,|x+8| + C}$$

5) $\int \frac{x^3+4}{x+8} dx$ =?

$$\int \frac{x^3+4}{x+8} = x^2 8x + 64 - \frac{508}{x+8}$$

$$\int (x^2 8x + 64 - \frac{508}{x+8}) dx$$

$$|x+8\,|In = \int 508x \frac{508}{x+8}$$

$$\int \frac{x^3+4}{x+8}\, dx$$

$$\frac{x^3}{3} - 4x^3 + 64x - 508\, In|x+8| + C$$

Problems(9.3): Evaluate the following integrals:

1) $\int \frac{x}{x^2+4x-5} dx$

$$= \frac{x}{(x+5)(x-1)} = \frac{\frac{5}{6}}{x+5} + \frac{\frac{1}{6}}{x-1}$$

$$\int \frac{\frac{5}{6}}{x+5} dx + \int \frac{\frac{1}{6}}{x-1} dx$$

$$\frac{5}{6}\, In\,|x+5| + \frac{1}{6}\, In\,|x-1| + C$$

2) $\int \frac{x}{x^2+2x-3}\,dx$

$$\frac{x}{(x-3)(x+1)} = \frac{\frac{3}{4}}{x-3} + \frac{\frac{1}{4}}{x-1}$$

$$\int \frac{\frac{3}{4}}{x-3}\,dx + \int \frac{\frac{1}{4}}{x-1}\,dx$$

$$\frac{3}{4}\,In\,|x-3| + \frac{1}{4}In\,|x-1| + C$$

3) $\int \frac{(x+1)}{x^2+4x-5}\,dx$

$$\frac{x+1}{(x+5)(x-1)} = \frac{\frac{2}{3}}{x+5} + \frac{\frac{1}{3}}{x-1}$$

$$\int \frac{\frac{2}{3}}{x+5}\,dx + \int \frac{\frac{1}{3}}{x-1}\,dx$$

$$\frac{2}{3}\,In\,|x+5| + \frac{1}{3}In\,|x-1| + C$$

4) $\int \frac{x^2}{x^2+2x-1}\,dx$

$$\frac{2x-1}{x^2+2x-1}$$

$$\int 1dx - \int \frac{2x+2-3}{x^2+2x-1}\,dx$$

$$x\,In\,|x^2+2x| + \frac{2}{2\sqrt{2}}In\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right| + C$$

5) $\int \frac{1}{x(x+1)^2}\,dx$

$\frac{1}{2}In\,|x+1| - In\,|x+1| + +C$

6) $\int \frac{1}{(x+1)(x^2+1)}\,dx$

$\frac{1}{2}In\,|x| - \frac{1}{4}In\,|x^2+1| + \frac{1}{2}\tan^{-1}(x) + C$

7) $\int \frac{1}{x(x^2+2x+1)}\,dx$

$In\,|x| - In\,|x+1| + \frac{1}{x+1} + C$

8) $\int \frac{sin\theta}{\cos^2\theta + cos\,\theta - 2}\,d\theta$

$\frac{1}{3}In\,|cos\theta + 2| - \frac{1}{3}In\,|cos\theta - 1| + C$

9) $\int \frac{e^t}{e^{2t}+3e^t+2}\,dt$

$In(e^t+1) - In\,(e^t+2) + C$

10) $\int \frac{(3x-7)}{(x-1)(x-2)(x-3)}\,dx$

$-2In\,|x-1| + In\,|x-2| + In\,|x-3| + c$

11) $\int \frac{\sin(x)}{cos^2x - 5\cos(x) + 4}\,dx$

$\frac{1}{3}In\,|\cos(x) - 4| - \frac{1}{3}In\,|\cos(x) - 1| + C$

12) $\int \frac{1}{x^2-6x+5}\,dx$

$\frac{1}{4}In\,|x-5| - \frac{1}{4}In\,|x-1| + C$

13) $\int \frac{x}{(x+2)\,(x+1)^2}\mathrm{dx}$

$$\frac{x}{(x+2)(x+1)^2} = \frac{1}{x+2} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

$$\int \left(\frac{-2}{x+2} + \frac{2}{x+1} - \frac{1}{(x+1)^2}\right) dx$$

$$= -2\,In|x+2| + 2In\,|x+1| + \frac{1}{x+1} + C$$

14) $\int \frac{2x-1}{(x^2-1)(x-2)}\,dx$

$$x^2 - 1 = (x-1)(x+1)$$

$$= \frac{1}{2}In\,|x-1| + \frac{1}{2}In\,|x+1| - In|x+2| + C$$

15) $\int \frac{x-1}{(x-1)(x^2+1)}\,dx$

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \int \left(\frac{-1}{x+1} + \frac{x}{x^2+1}\right) dx$$

$$= -In\,|\,x+1| + \frac{1}{2}\;\; In(x^2+1) + C$$

16) $\int \frac{5x-3}{x^2-2x-3}\,dx$

$$x^2 - 2x - 3 = (x-3)(x+1)$$

$$= \frac{1}{x-3} + \frac{1}{x+1}$$

$$2In\,|x-3| + 3In\,|x+1| + C$$

17) $\int \frac{\pi^2}{x^2-9}\, dx$

$$\int (1 + \frac{9}{x^2+9})\, dx$$

$$= x + 9 \int \frac{1}{(x-3)(x+3)}\, dx$$

$$= x + \frac{9}{6}\, In\, \left|\frac{x-3}{x+3}\right| + C$$

$$= x + \frac{3}{2}\, In\, \left|\frac{x-3}{x+3}\right| + C$$

18) $\int \frac{1}{(x-1)^2}\, dx$

$$\int (x-1)^{-2}\, dx$$

$$= -\frac{1}{x-1} + C$$

19) $\int \frac{x}{x^2-4x-5}\, dx$

$$x^2 - 4x - 5 = (x-5)(x+1)$$

$$= \int \frac{5}{6(x-5)} + \frac{1}{6(x+1)}\, dx$$

$$= \frac{5}{6}\, In\, |x-5| + \frac{1}{6} In + C$$

20) $\int \frac{(2x+41)}{x^2-5x-14}\, dx$

$$x^2 + 5x - 14 = (x+7)(x-2)$$

$$= 3In\, |x+7| - In\, |x-2| + C$$

21) $\int \frac{x}{(x-1)(x+1)^2}\, dx$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{1}{4} In\,|x-1| + \frac{3}{4}\, In\,|x+1| + \frac{1}{2(x+1)} + C$$

٩,٤ Integration by Trigonometric Substitution:

In this section, we see how to integrate expressions like and

$\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, and $\sqrt{u^2 - a^2}$

depending on the function we need to integrate, we substitute one of the following trigonometric expressions to simplify the integration:

- For $\sqrt{a^2 - u^2}$, use $u = a\,sin\theta$
- For $\sqrt{a^2 + u^2}$, use $u = a\,tan\theta$
- For $\sqrt{u^2 - a^2}$, use $u = a\,sec\theta$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{9-x^2}} = ?$

$\because \sqrt{9-x^2} \equiv \sqrt{a^2-u^2}$ *we use* $\boxed{u = a\sin\theta}$

$\because a = 3 \text{ and } u = x \Longrightarrow \boxed{x = 3\sin\theta}$

$\Longrightarrow \boxed{\theta = \sin^{-1}\left(\frac{x}{3}\right)}$ *and* $\boxed{dx = 3\cos\theta d\theta}$

$\therefore \sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} =$

$= \sqrt{9(1-\sin^2\theta)} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta = \boxed{3\cos\theta}$

$\therefore \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta d\theta}{3\cos\theta} = \int d\theta = \theta + C = \boxed{\sin^{-1}\left(\frac{x}{3}\right) + C}$

2) $\int \frac{dx}{\sqrt{25+x^2}} = ?$

$\because \sqrt{25+x^2} \equiv \sqrt{a^2+u^2}$ *we use* $\boxed{u = a\tan\theta}$

$\because a = 5 \text{ and } u = x \Longrightarrow \boxed{x = 5\tan\theta}$

$\Longrightarrow \boxed{\theta = \tan^{-1}\left(\frac{x}{5}\right)}$ *and* $\boxed{dx = 5\sec\theta d\theta}$

$\therefore \sqrt{25+x^2} = \sqrt{25-(5\tan\theta)^2} = \sqrt{25+25\tan^2\theta}$

$= \sqrt{25(1+\tan^2\theta)} = 5\sqrt{1+\tan^2\theta} = 5\sqrt{\sec^2\theta} = 5\sec\theta = \boxed{5\sec\theta}$

$\therefore \int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5\sec^2\theta d\theta}{5\sec\theta} = \int \sec\theta d\theta$

$= In\left|sec\,\theta + ta\,n\,\theta\right| + C = \boxed{In\left|\frac{\sqrt{25+x^2}}{5} + \frac{5}{x}\right| + C}$

3) $\int \frac{dx}{x\sqrt{x^2-16}} = ?$

$\because \sqrt{x^2-16} \equiv \sqrt{u^2-a^2}$ we use $\boxed{u = a\,sec\theta}$

$\because a = 4$ and $u = x \implies \boxed{x = 4sec\theta}$

$\implies \boxed{\theta = sec^{-1}\left(\frac{x}{4}\right)}$ and $dx = \boxed{4\sec\theta\tan\theta\,d\theta}$

$\therefore \sqrt{x^2-16} = \sqrt{(4\sec\theta)^2-16} = \sqrt{16\,sec^2\,\theta-16}$

$= \sqrt{16(sec^2\theta-1)} = 4\sqrt{sec^2-1} = 4\sqrt{tan^2\,\theta} = 4\tan\theta$

$\therefore \int \frac{dx}{x\sqrt{x^2-16}} = \int \frac{4\sec\theta\tan\theta d\theta}{4sec\theta * 4\tan\theta} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C = \boxed{\frac{1}{4}sec^{-1}\left(\frac{x}{4}\right) + C}$

Problems(9.4): Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1+4x^2}}$

$$\int \frac{dx}{\sqrt{1-a^2-x^2}} = \frac{1}{a}\sin^{-1}(ax) + C$$

$$\int \frac{dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sin^{-1}(2x) + C$$

2) $\int \frac{dx}{\sqrt{4+x^2}} = In\,|x + \sqrt{4+x^2}\,| + c$

3) $\int \frac{dx}{\sqrt{4-(x-1)^2}} =$

4=x-1=0 d = dx

$$= \int \frac{du}{\sqrt{x-1}} = \sin^{-1}(\frac{4}{2}) = \sin^{-1}(\frac{x-1}{2}) + C$$

4) $\int \frac{xdx}{4+x^2}$, $du = 2xdx$

$Xdx = du$

$$\int \frac{1}{u} \cdot \frac{1}{2}\, du = \frac{1}{2} \int \frac{du}{u}$$

$$= \frac{1}{2}\, In\,|4| + C$$

$$= \frac{1}{2}\, In(4 + x^2) + C$$

5) $\int \frac{(x+1)dx}{\sqrt{4-x^2}}$

$$\int \frac{xdx}{\sqrt{4 - x^2}} + \frac{dx}{\sqrt{4 - x^2}}$$

$$\int \frac{xdx}{\sqrt{4 - x^2}}$$

$$4 = 4 - x^2 = du\ = -2xdx$$

$$Xdx = -\frac{1}{2}\, du$$

$$= -\frac{1}{2} \int \frac{du}{\sqrt{4}} = \frac{1}{2}\left(2\sqrt{4}\right) = -\sqrt{4 - x^2}$$

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1}(\frac{x}{2})$$

$$= -\sqrt{4 - x^2} + \sin^{-1}(\frac{x}{2}) + C$$

6) $\int \frac{dx}{\sqrt{(9-x^2)^{\frac{3}{2}}}}$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^{\frac{3}{2}}}} = \frac{x}{a^2\sqrt{a^{2-} - x^2}} + C$$

a=3

$$= \frac{x}{9\sqrt{9-x^2}} + C$$

7) $\int \frac{xdx}{\sqrt{16-x^2}}$

$4 = 16 - x^2 \rightarrow du = -2xdx$

$Xdx = -\frac{1}{2}\, du$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{4}} = \sqrt{16-x^2} + C$$

8) $\int \frac{dx}{\sqrt{(x^2+1)^2}}$

$\sqrt{(x^2+1)^2} = x^2 + 1$

$$= \int \frac{dx}{x^2+1} = \tan^{-1}(x) + C$$