



جامعة تكريت

كلية التربية للبنات

قسم الرياضيات

تفاضل وتكامل

المسافة تحت المنحني

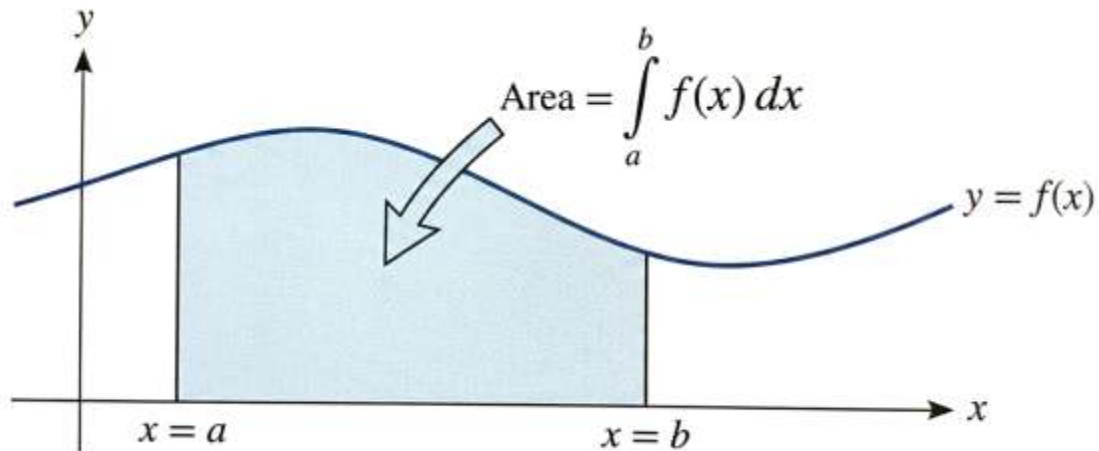
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### 10.3 Area Under a Curve

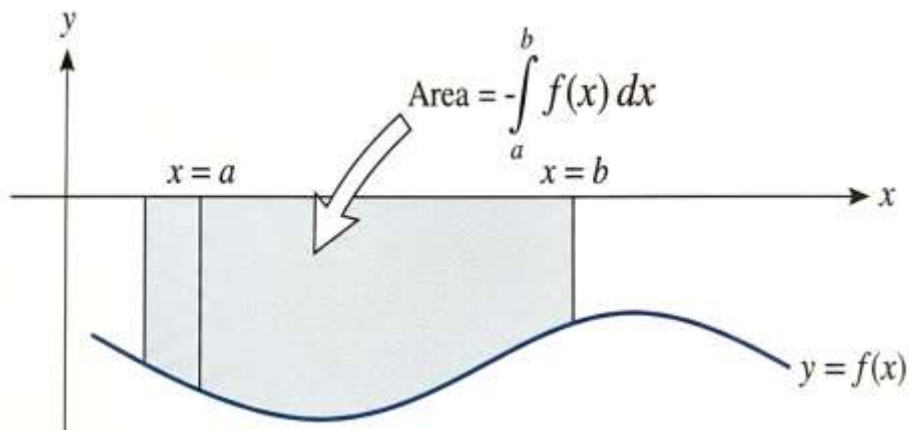
The area under a curve between two points can be found by doing a definite integral between the two points. To find the area under the curve

$y = f(x)$  between  $x = a$  and  $x = b$ , integrate  $y = f(x)$  between the limits of  $a$  and  $b$ .



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**Remark:** If the area is above  $x$ -axis, then the area is positive, and if the area under the  $x$ -axis, the area is negative, so we should change the sign to positive value by adding a negative sign or by taking the absolute value.



**Remark:** To avoid the negative value, we will take the absolute value:

$$Area = \left| \int_{x=a}^{x=b} f(x) dx \right|$$

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**Example (1):** Find the area bounded by  $y = x^2$  and  $x = 1$  and  $x = 3$ ?

**Solution:**

$$\begin{aligned} Area &= \left| \int_{x=1}^{x=3} x^2 dx \right| \\ &= \left| \left[ \frac{x^3}{3} \right]_{x=1}^{x=3} \right| \\ &= \left| \frac{3^3}{3} - \frac{1^3}{3} \right| \\ &= \left| 9 - \frac{1}{3} \right| = 8\frac{2}{3} \text{ unit}^2 \end{aligned}$$

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**Example (2):** Find the total area between the curve  $y = x^3$  and  $x = -2$  and  $x = 2$ ?

**Solution:** If we simply integrated  $x^3$  between  $x = -2$  and  $x = 2$ , we would get:

$$\int_{-2}^2 x^3 dx = \left[ \frac{x^4}{4} \right]_{-2}^2 = \frac{16}{4} - \frac{16}{4} = 0$$

So, instead we have to split the graph up and do two separate integrals:

$$A_1 = \left| \int_{-2}^0 x^3 dx \right| = \left| \left[ \frac{x^4}{4} \right]_{-2}^0 \right| = |0 - 4| = 4$$

$$A_2 = \left| \int_0^2 x^3 dx \right| = \left| \left[ \frac{x^4}{4} \right]_0^2 \right| = |4 - 0| = 4$$

Hence,  $Area = A_1 + A_2 = 4 + 4 = 8 \text{ unit}^2$

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**Example (3):** Find the area bounded by the line  $x + y = 1$  and the coordinate axes?

**Solution:** *Note: The handwriting in the solution provided in the image changes the equation to  $x + y = 2$ . Following the image's calculation:*

$$x + y = 2 \Rightarrow y = 2 - x$$

$$x = 0 \Rightarrow y = 2 \Rightarrow (0,2)$$

$$y = 0 \Rightarrow x = 2 \Rightarrow (2,0)$$

$$Area = \left| \int_0^2 (2 - x) dx \right| = \left| \left[ 2x - \frac{x^2}{2} \right]_0^2 \right| = |(4 - 2) - 0| = 2 \text{ unit}^2$$

**Solution for the original question ( $x + y = 1$ ):**

$$y = 1 - x$$

$$Area = \left| \int_0^1 (1 - x) dx \right| = \left| \left[ x - \frac{x^2}{2} \right]_0^1 \right| = \left| 1 - \frac{1}{2} \right| = \frac{1}{2} \text{ unit}^2$$

**Another way:**

$$\because x + y = 2 \Rightarrow x = 2 - y$$

$$x = 0 \Rightarrow y = 2 \Rightarrow (0,2)$$

$$Area = \left| \int_{y=0}^{y=2} (2 - y) dy \right|$$

$$= \left| \left[ 2y - \frac{y^2}{2} \right]_{y=0}^{y=2} \right|$$

$$= \left| \left( 2(2) - \frac{2^2}{2} \right) - (0 - 0) \right|$$

$$= |4 - 2| = 2 \text{ unit}^2$$

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### **Problems (10.3):**

1) Find the total area bounded by the curve  $y = x^3 - 4x$  and  $x$ -axis.

$$x^3 - 4x = 0 \Rightarrow x(x - 2)(x + 2) = 0 \Rightarrow x = 0, 2, -2$$

$$Area = \left| \int_{-2}^0 (x^3 - 4x) dx \right| + \left| \int_0^2 (x^3 - 4x) dx \right|$$

$$= \left| \left[ \frac{x^4}{4} - 2x^2 \right]_{-2}^0 \right| + \left| \left[ \frac{x^4}{4} - 2x^2 \right]_0^2 \right| = |-(4 - 8)| + |(4 - 8)| = 4 + 4$$
$$= 8$$

2) Find the area bounded by  $y = 4 - x^2$  and  $x$ -axis.

$$4 - x^2 = 0 \Rightarrow x = \pm 2$$

$$Area = \left| \int_{-2}^2 (4 - x^2) dx \right| = \left| \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 \right| = \left| \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) \right|$$
$$= \frac{32}{3}$$

- 3) Find the area bounded by  $x = y^2 - y^3$  and  $y$ -axis.

$$y^2 - y^3 = 0 \Rightarrow y^2(1 - y) = 0 \Rightarrow y = 0, 1$$

$$Area = \left| \int_0^1 (y^2 - y^3) dy \right| = \left| \left[ \frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 \right| = \left| \frac{1}{3} - \frac{1}{4} \right| = \frac{1}{12}$$

- 4) Find the area bounded by  $\sqrt{x} + \sqrt{y} = 1$  and the two axes.

$$\sqrt{y} = 1 - \sqrt{x} \Rightarrow y = (1 - \sqrt{x})^2 = 1 - 2\sqrt{x} + x \quad (\text{Limits: } x = 0 \text{ to } x = 1)$$

$$Area = \left| \int_0^1 (1 - 2x^{1/2} + x) dx \right| = \left| \left[ x - \frac{4}{3}x^{3/2} + \frac{x^2}{2} \right]_0^1 \right| = \left| 1 - \frac{4}{3} + \frac{1}{2} \right| = \frac{1}{6}$$

- 5) Prove that the area under one curve of  $y = \sin(x)$  equals to 2 units<sup>2</sup>.

One arch of  $\sin(x)$  is from  $x = 0$  to  $x = \pi$ :

$$Area = \int_0^\pi \sin(x) dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) + 1 = 2$$

- 6) Find the area bounded by  $y = x^2 - 4x$  and  $x$ -axis.

$$x^2 - 4x = 0 \Rightarrow x(x - 4) = 0 \Rightarrow x = 0, 4$$

$$Area = \left| \int_0^4 (x^2 - 4x) dx \right| = \left| \left[ \frac{x^3}{3} - 2x^2 \right]_0^4 \right| = \left| \frac{64}{3} - 32 \right| = \left| -\frac{32}{3} \right| = \frac{32}{3}$$

- 7) Find the area bounded by  $x = 8 - 2y - y^2$  and  $y$ -axis.

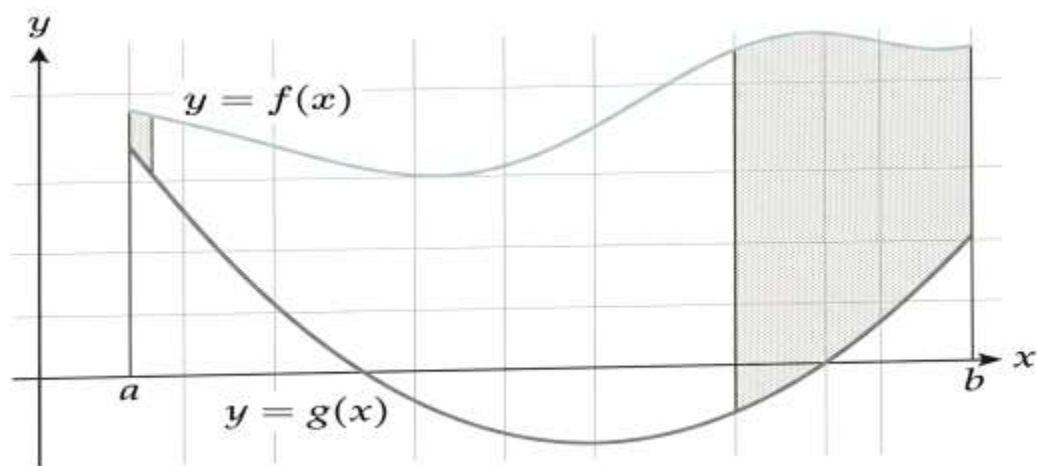
$$8 - 2y - y^2 = 0 \Rightarrow -(y + 4)(y - 2) = 0 \Rightarrow y = -4, 2$$

$$Area = \left| \int_{-4}^2 (8 - 2y - y^2) dy \right| = \left| \left[ 8y - y^2 - \frac{y^3}{3} \right]_{-4}^2 \right| = \left| \left( 16 - 4 - \frac{8}{3} \right) - \left( -32 - 16 + \frac{64}{3} \right) \right| = \left| \frac{28}{3} - \left( -\frac{80}{3} \right) \right| = \frac{108}{3} = 36$$

## 10.4 Area Between Two Curves

In this section we are going to look at finding the area between two curves. There are actually two cases that we are going to be looking at.

In the first case we want to determine the area between  $y = f(x)$  and  $y = g(x)$  on the interval  $[a, b]$ . We are also going to assume that  $f(x) \geq g(x)$ . Take a look at the following sketch to get an idea of what we're initially going to look at.

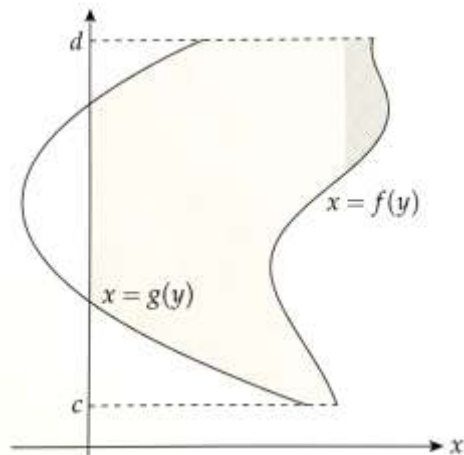


$$\text{Area} = \left| \int_{x=a}^{x=b} [f(x) - g(x)] dx \right|$$

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The second case is almost identical to the first case. Here we are going to determine the area between  $x = f(y)$  and  $x = g(y)$  on the interval  $[c, d]$  with  $f(y) \geq g(y)$ .

$$\text{Area} = \left| \int_{y=c}^{y=d} [f(y) - g(y)] dy \right|$$



**Example (1):** Find the area between the curve  $y = 2 - x^2$  and the line  $y = -x$ ?

$$y_1 = y_2 \Rightarrow 2 - x^2 = -x$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ and } x = -1$$

$$\text{Area} = \left| \int_{x=-1}^{x=2} [f(x) - g(x)] dx \right|$$

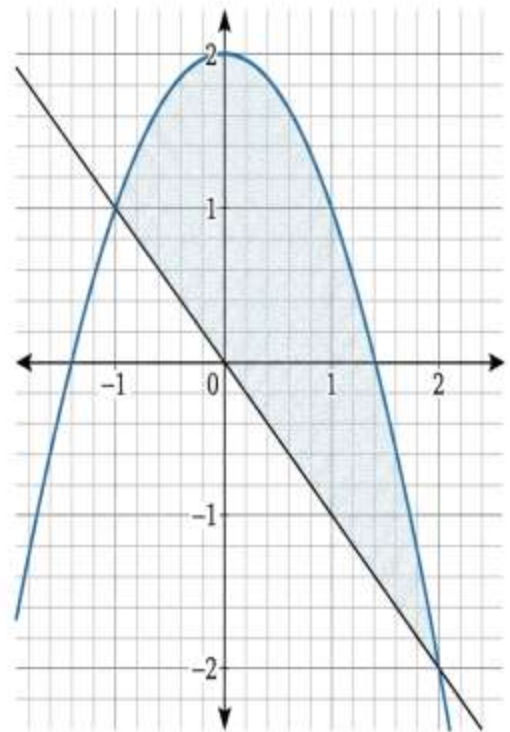
$$= \left| \int_{x=-1}^{x=2} [(2 - x^2) - (-x)] dx \right|$$

$$= \left| \int_{x=-1}^{x=2} [2 - x^2 + x] dx \right|$$

$$= \left| \left[ 2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{x=-1}^{x=2} \right|$$

$$= \left| \left( 4 - \frac{8}{3} + 2 \right) - \left( -2 + \frac{1}{3} + \frac{1}{2} \right) \right|$$

$$= \left| \frac{10}{3} - \left( -\frac{7}{6} \right) \right| = \left| \frac{20 + 7}{6} \right| = \frac{27}{6} \text{ units}^2$$



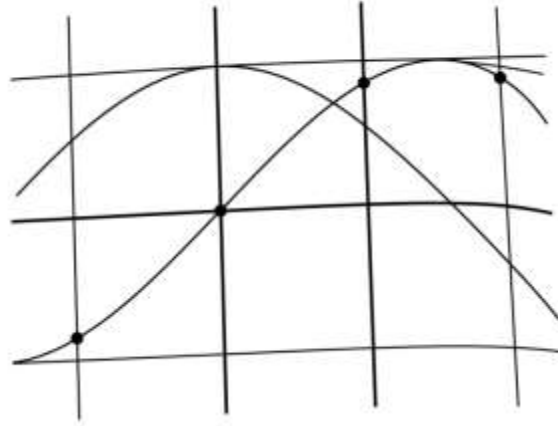
**Example (2):** Find the area of the triangular shaped region in the first quarter bounded by the y-axis and the curves  $y = \sin(x)$  and  $y = \cos(x)$ ?

**Solution:**

$$y_1 = y_2$$

$$\rightarrow \cos(x) = \sin(x)$$

$$\rightarrow x = \frac{\pi}{4}$$



$$\text{Area} = \left| \int_{x=0}^{x=\frac{\pi}{4}} [f(x) - g(x)] dx \right|$$

$$= \left| \int_{x=0}^{x=\frac{\pi}{4}} [\cos(x) - \sin(x)] dx \right|$$

$$= \left| [\sin(x) + \cos(x)]_{x=0}^{x=\frac{\pi}{4}} \right|$$

$$= \left| \left( \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \right) - (\sin(0) + \cos(0)) \right|$$

$$= \left| \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1) \right|$$

$$= \sqrt{2} - 1 \text{ units}^2$$

**Example (3):** Find the area bounded between the two curves  $y = x^2$  and  $y = |x|$ ?

**Solution:**

$$y_1 = y_2 \rightarrow |x| = x^2$$

$$\rightarrow \sqrt{x^2} = x^2$$

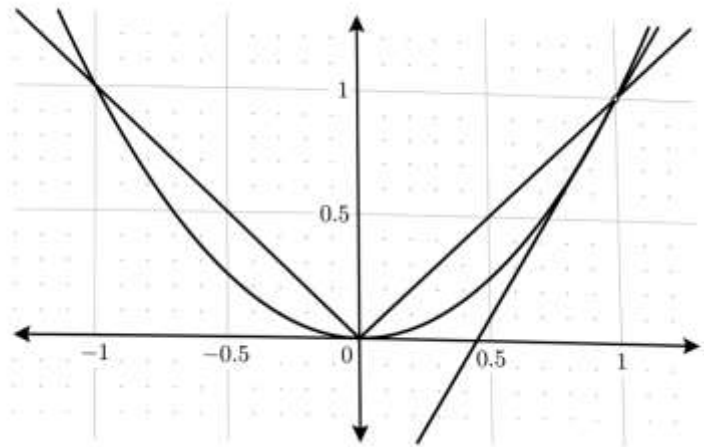
$$\rightarrow x^2 = x^4 \rightarrow x^2 - x^4 = 0$$

$$\rightarrow x^2(1 - x^2) = 0$$

$$\rightarrow x^2(1 - x)(1 + x) = 0$$

$$x^2 = 0 \Rightarrow x = 0 \rightarrow (0,0)$$

$$(x + 1) = 0 \Rightarrow x = -1 \rightarrow (-1,1)$$



$$\begin{aligned} A1 &= \left| \int_{x=-1}^{x=0} [f(x) - g(x)] dx \right| = \left| \int_{-1}^0 [-x - x^2] dx \right| = \left| \left[ -\frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^0 \right| \\ &= \left| 0 - \left( -\frac{1}{2} - \frac{-1}{3} \right) \right| = \left| \frac{1}{2} - \frac{1}{3} \right| = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} A2 &= \left| \int_{x=0}^{x=1} [f(x) - g(x)] dx \right| = \left| \int_0^1 [x - x^2] dx \right| = \left| \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \right| \\ &= \left| \left( \frac{1}{2} - \frac{1}{3} \right) - 0 \right| = \left| \frac{1}{2} - \frac{1}{3} \right| = \frac{1}{6} \end{aligned}$$

$$Area = A1 + A2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \text{ units}^2$$

**Example (4):** Find the area bounded between curve  $y = \frac{1}{x^2}$  and the two lines  $y = 1$  and  $y = 3$ ?

**Solution:**

$$y = \frac{1}{x^2}$$

$$\rightarrow x^2 = \frac{1}{y}$$

$$\rightarrow x = \mp \frac{1}{\sqrt{y}}$$

$$\text{Area} = 2 \left| \int_{y=1}^{y=3} [f(y) - g(y)] dy \right|$$

$$= 2 \left| \int_1^3 \frac{1}{\sqrt{y}} dy \right|$$

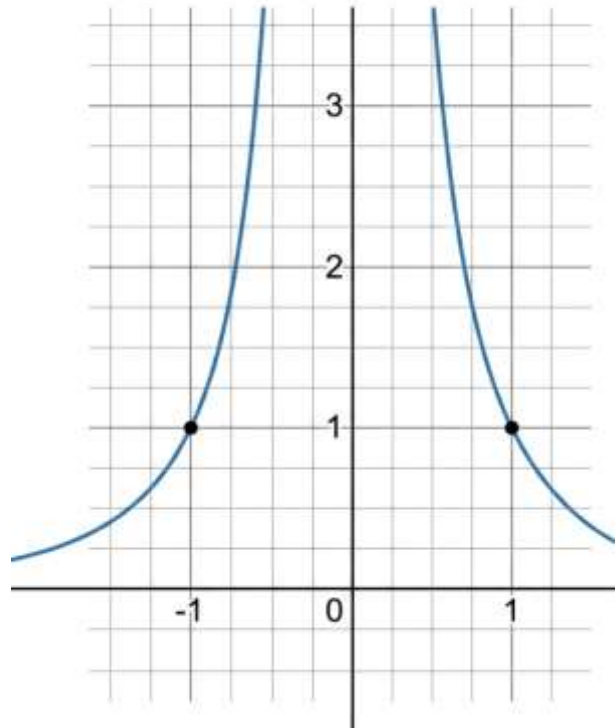
$$= 2 \left| \int_1^3 y^{-1/2} dy \right|$$

$$= 2 \left| \left[ \frac{y^{1/2}}{1/2} \right]_1^3 \right|$$

$$= 2 \left| 2\sqrt{y} \right|_1^3$$

$$= 4(\sqrt{3} - \sqrt{1})$$

$$= 4(\sqrt{3} - 1) \text{ units}^2$$



### **Problems (10.4):**

- 1) Find the area bounded by the curve  $y = \sqrt{x}$  and the line  $y = x$ :**

$$\sqrt{x} = x \Rightarrow x = x^2 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, 1.$$

$$\text{Area} = \int_0^1 (\sqrt{x} - x) dx = \left[ \frac{2}{3} x^{3/2} - \frac{1}{2} x^2 \right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ units}^2$$

- 2) Find the area bounded by the curve  $y = x^3$  and lines  $x = -1$  and  $x = -3$ :**

$$\text{Area} = \left| \int_{-3}^{-1} x^3 dx \right| = \left| \left[ \frac{x^4}{4} \right]_{-3}^{-1} \right| = \left| \frac{1}{4} - \frac{81}{4} \right| = |-20| = 20 \text{ units}^2$$

- 3) Find the area bounded by the curve  $y = x - x^2$  and  $y = x^2 - x$ :**

$$x - x^2 = x^2 - x \Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x - 1) = 0 \Rightarrow x = 0, 1.$$

$$\text{Area} = \int_0^1 [(x - x^2) - (x^2 - x)] dx = \int_0^1 (2x - 2x^2) dx$$

$$= \left[ x^2 - \frac{2}{3} x^3 \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \text{ units}^2$$

- 4) Find the area bounded by the curve  $x = 4y - y^2 - 3$  and line  $x = -3$ :**

$$-3 = 4y - y^2 - 3 \Rightarrow y^2 - 4y = 0 \Rightarrow y(y - 4) = 0 \Rightarrow y = 0, 4.$$

$$\text{Area} = \int_0^4 [(4y - y^2 - 3) - (-3)] dy = \int_0^4 (4y - y^2) dy =$$

$$\left[ 2y^2 - \frac{1}{3} y^3 \right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3} \text{ units}^2$$

- 5) Find the area bounded by the curve  $y = x^3$  and  $y = x$  in the first quarter:**

$$x^3 = x \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = 0, 1 \text{ (first quarter).}$$

$$\text{Area} = \int_0^1 (x - x^3) dx = \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ units}^2$$

- 6) Find the area bounded by the curve  $y = \frac{1}{x}$  and lines  $y = 2$  and  $y = 3$ :**

$$x = \frac{1}{y}. \text{Area} = \int_2^3 \frac{1}{y} dy = [\ln|y|]_2^3 = \ln(3) - \ln(2) = \ln(1.5) \text{ units}^2$$

- 7) Find the area bounded by the curve  $y = e^x$ ,  $y = e^{-x}$  and lines  $y = 2$  and  $y = 4$ :**

Intersection of curves is at  $x = 0$ . Rewrite as  $x = \ln(y)$  and  $x = -\ln(y)$ .

$$\begin{aligned} \text{Area} &= \int_2^4 [\ln(y) - (-\ln(y))] dy = 2 \int_2^4 \ln(y) dy = 2[y \ln(y) - y]_2^4 = \\ &2[(4 \ln 4 - 4) - (2 \ln 2 - 2)] = 2[8 \ln 2 - 4 - 2 \ln 2 + 2] = 12 \ln 2 - 4 \text{ units}^2 \end{aligned}$$

- 8) Find the area bounded by the curve  $y = x^2 + 2$  and  $y = x + 5$ :**

$$\text{Intersection: } x^2 + 2 = x + 5 \Rightarrow x^2 - x - 3 = 0 \Rightarrow x = \frac{1 \pm \sqrt{13}}{2}. \text{Area} =$$

$$\int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} [(x+5) - (x^2+2)] dx = \int_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} (-x^2 + x + 3) dx =$$

$$\left[ -\frac{x^3}{3} + \frac{x^2}{2} + 3x \right]_{\frac{1-\sqrt{13}}{2}}^{\frac{1+\sqrt{13}}{2}} = \frac{13\sqrt{13}}{6} \text{ units}^2$$