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قسم الرياضيات

تفاضل وتكامل

تكامل الدوال المثلثية العكسية

ا.م.د.هبه هاني عبدالله

hiba.h.a.83@tu.edu.iq

8.3 Integrals of Inverse Trigonometric Functions

We can derive all the integration forms from our derivatives forms as follows:

1) $\frac{d}{du}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}} \Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(u) + C$

2) $\frac{d}{du}\cos^{-1}(u) = -\frac{1}{\sqrt{1-u^2}} \Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = -\cos^{-1}(u) + C$

3) $\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2} \Rightarrow \int \frac{1}{1+u^2}du = \tan^{-1}(u) + C$

4) $\frac{d}{du}\cot^{-1}(u) = -\frac{1}{1+u^2} \Rightarrow \int \frac{1}{1+u^2}du = -\cot^{-1}(u) + C$

5) $\frac{d}{du}\sec^{-1}(u) = \frac{1}{|u|\sqrt{u^2-1}} \Rightarrow \int \frac{1}{|u|\sqrt{u^2-1}}du = \sec^{-1}(u) + C$

6) $\frac{d}{du}\csc^{-1}(u) = -\frac{1}{|u|\sqrt{u^2-1}} \Rightarrow \int \frac{1}{|u|\sqrt{u^2-1}}du = -\csc^{-1}(u) + C$

Examples: Evaluate the following integrals:

1) $\int \frac{dx}{\sqrt{1-4x^2}}$

$= \frac{1}{2}\sin^{-1}(2x) + C$ or $-\frac{1}{2}\cos^{-1}(2x) + C$

2) $\int \frac{dt}{1+t^2}$

$= \tan^{-1}(t) + C$ or $-\cot^{-1}(t) + C$

3) $\int \frac{dx}{x\sqrt{4x^2-1}}$

$= \int \frac{2dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}|2x| + C$ or $-\csc^{-1}|2x| + C$

4) $\int \frac{-dx}{\sqrt{4-25x^2}} = \int \frac{-dx}{\sqrt{4\left(1-\frac{25}{4}x^2\right)}} = \int \frac{-dx}{2\sqrt{1-\left(\frac{5}{2}x\right)^2}}$

$= -\frac{1}{2}\cdot\frac{2}{5}\int \frac{\frac{5}{2}dx}{\sqrt{1-\left(\frac{5}{2}x\right)^2}}$

$= -\frac{1}{5}\sin^{-1}\left(\frac{5}{2}x\right) + C$ or $\frac{1}{5}\cos^{-1}\left(\frac{5}{2}x\right) + C$

5) $\int \frac{\cos(x)dx}{\sqrt{1-\sin^2(x)}}$

$= \sin^{-1}\big(\sin(x)\big) + C$

$= x + C$

6) $\int \frac{\tan^{-1}(x)}{1+x^2}dx$

$= \int \tan^{-1}(x) \cdot \left(\frac{dx}{1+x^2}\right)$

$= \frac{\big(\tan^{-1}(x)\big)^2}{2} + C$

7) $\int \frac{\sqrt{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx$

$= \big(\sec^{-1}(x)\big)^{\frac{1}{2}} \cdot \left(\frac{1}{x\sqrt{x^2-1}}\right)dx$

$= \frac{\big(\sec^{-1}(x)\big)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\big(\sec^{-1}(x)\big)^{\frac{3}{2}} + C$

Problems (8.3) :

1) $\int \frac{1}{\sqrt{1-(2+9z)^2}}dz = \frac{1}{9}sin^{-1}(2+9z) + C$

2) $\int d\big(csc^{-1}(t)\big) = csc^{-1}(t) + C$

3) $\int \frac{x}{1+16x^4}dx = \frac{1}{8}tan^{-1}(4x^2) + C$

4) $\int \left(cos^{\frac{1}{2}}(x)sin(x) - \frac{13}{1+x^2}\right)dx = -\frac{2}{3}cos^{\frac{3}{2}}(x) - 13tan^{-1}(x) + C$

5) $\int \frac{x}{9x^4+x^2}dx = \int \frac{1}{9x^3+x}dx \Longrightarrow \frac{1}{2}ln|x^2| - \frac{1}{2}ln|9x^2+1| + C$

6) $\int \frac{3}{16z^2+4}dz = \frac{3}{8}tan^{-1}(2z) + C$

7) $\int \frac{sec^2(x)}{\sqrt{1-tan^2(x)}}dx = sin^{-1}\big(tan(x)\big) + C$

8) $\int \frac{\left(sin^{-1}(x)\right)^2}{\sqrt{1-x^2}}dx = \frac{\left(sin^{-1}(x)\right)^3}{3} + C$

9) $\int \frac{sin\left(tan^{-1}(x)\right)}{1+x^2}dx = -cos\left(tan^{-1}(x)\right) + C$

10) $\int \frac{1}{1+25x^2}dx = \frac{1}{5}tan^{-1}(5x) + C$

11) $\int \frac{-1}{\sqrt{1-16w^2}}dw = -\frac{1}{4}sin^{-1}(4w) + C$

12) $\int d\left(sec^{-1}(t)\right) = sec^{-1}(t) + C$

13) $\int \frac{-3t}{9t^4+9}dt = -\frac{1}{6}tan^{-1}(t^2) + C$

14) $\int \left(cos^3(2x)sin(2x) - \frac{x^3}{\pi}\right)dx = -\frac{1}{8}cos^4(2x) - \frac{x^4}{4\pi} + C$

15) $\int \frac{1}{25+t^2}dt = \frac{1}{5}tan^{-1}\left(\frac{t}{5}\right) + C$

16) $\int \frac{1}{81w^2+9}dw = \frac{1}{27}tan^{-1}(3w) + C$

17) $\int \frac{sec^2(x)}{\sqrt{1-tan^2(x)}}dx = sin^{-1}\left(tan(x)\right) + C$

18) $\int \frac{\left(cos^{-1}(2z)\right)^2}{\sqrt{1-4z^2}}dz = -\frac{1}{6}\left(cos^{-1}(2z)\right)^3 + C$

19) $\int \frac{sin\left(sin^{-1}(3x)\right)}{\sqrt{1-9x^2}}dx = \int \frac{3x}{\sqrt{1-9x^2}}dx = -\frac{1}{3}\sqrt{1-9x^2} + C$

20) $\int \frac{-\pi}{3+27t^2}dt = -\frac{\pi}{9}tan^{-1}(3t) + C$

8.4 Integrals of Logarithmic Functions

$$\frac{d}{du}\ln(u) = \frac{1}{u}du \Longrightarrow \int \frac{1}{u}du = \ln|u| + C, \quad u \neq 0$$

Examples: Evaluate the following integrals:

1) $\int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2ln|x| + C$

2) $\int \left(\frac{3}{x^2} + \frac{5}{x}\right)dx = 3\int x^{-2}dx + 5\int \frac{1}{x}dx$

$= 3\frac{x^{-1}}{-1} + 5ln|x| + C = \frac{-3}{x} + 5ln|x| + C$

3) $\int \frac{x}{(2x^2+3)}dx = \frac{1}{4}\int \frac{4x}{(2x^2+3)}dx = \frac{1}{4}ln|2x^2 + 3| + C$

4) $\int \frac{ln(x)}{x}dx = \int ln(x) \cdot \frac{1}{x}dx = \frac{(ln(x))^2}{2} + C$

5) $\int \frac{1}{xln(x)}dx = \int \frac{\frac{1}{x}}{ln(x)}dx = ln|ln(x)| + C$

6) $\int \frac{e^x}{1+2e^x}dx = \frac{1}{2}\int \frac{2e^x}{1+2e^x}dx = \frac{1}{2}ln|1 + 2e^x| + C$

7) $\int \frac{sec^2(x)}{tan(x)}dx = ln|tan(x)| + C$

8) $\int \frac{sec(2x)tan(2x)}{sec(2x)}dx = \frac{1}{2}\int \frac{2sec(2x)tan(2x)}{sec(2x)}dx = ln|sec(2x)| + C$

9) $\int tan(u)du = -\int \frac{-sin(u)}{cos(u)}du = -ln|cos(u)| + C$

10) $\int cot(u)du = \int \frac{cos(u)}{sin(u)}du = ln|sin(u)| + C$

11) $\int sec(u)du = \int sec(u)\frac{sec(u)+tan(u)}{sec(u)+tan(u)}du = ln|sec(u)+tan(u)|+C$

$= \int sec(u)\cdot\frac{(sec(u)+tan(u))}{(sec(u)+tan(u))}du = \int \frac{sec^2(u)+sec(u)tan(u)}{tan(u)+sec(u)}du$

$= ln|tan(u)+sec(u)|+C$

12) $\int csc(u)du = \int csc(u)\cdot\frac{(csc(u)+cot(u))}{(csc(u)+cot(u))}du = -\int csc(u)\cdot$

$\frac{(-csc(u)-cot(u))}{(csc(u)+cot(u))}du = -\int \frac{csc^2(u)+csc(u)cot(u)}{cot(u)+csc(u)}du$

$= -ln|cot(u)+csc(u)|+C$

Problems (8.4): Evaluate the following integrals:

1) $\int \frac{1}{x-3}dx = ln|x-3|+C$

2) $\int \frac{dx}{x\cdot ln^5(x)} = \int \big(ln(x)\big)^{-5}\cdot\frac{1}{x}dx = \frac{(ln(x))^{-4}}{-4}+C$

3) $\int \frac{xdx}{4x^2+1} = \frac{1}{8}\int \frac{8x}{4x^2+1}dx = \frac{1}{8}ln(4x^2+1)+C$

4) $\int \frac{2x-5}{x}dx = \int \left(2-\frac{5}{x}\right)dx = 2x-5ln|x|+C$

5) $\int \frac{cos(x)}{sin(x)}dx = ln|sin(x)|+C$

6) $\int \frac{sin(x)}{2-cos(x)}dx = ln|2-cos(x)|+C$

7) $\int \frac{x}{1-x^2}dx = -\frac{1}{2}\int \frac{-2x}{1-x^2}dx = -\frac{1}{2}ln|1-x^2|+C$

8) $\int \frac{ln^2(x)}{x}dx = \int \left(ln(x)\right)^2 \cdot \frac{1}{x}dx = \frac{ln^3(x)}{3} + C$

9) $\int \frac{5dx}{\sqrt{1-9x^2}} = 5\int \frac{1}{\sqrt{1-(3x)^2}}dx = \frac{5}{3}sin^{-1}(3x) + C$

10) $\int \frac{x+10}{x^2}dx = \int \left(\frac{1}{x} + 10x^{-2}\right)dx = ln|x| - \frac{10}{x} + C$

11) $\int \frac{y^2+2y+1}{(y+1)^3}dy = \int \frac{(y+1)^2}{(y+1)^3}dy = \int \frac{1}{y+1}dy = ln|y+1| + C$

12) $\int \frac{dx}{2-3x} = -\frac{1}{3}ln|2-3x| + C$

13) $\int \frac{x^2}{4-x^3}dx = -\frac{1}{3}ln|4-x^3| + C$

14) $\int \frac{x}{x+1}dx = \int \frac{x+1-1}{x+1}dx = \int \left(1 - \frac{1}{x+1}\right)dx = x - ln|x+1| + C$

15) $\int \left(\frac{x^{\frac{3}{2}}}{4x^{\frac{5}{2}}} + 6\right)dx = \int \left(\frac{1}{4x} + 6\right)dx = \frac{1}{4}ln|x| + 6x + C$

16) $\int \frac{sin(\theta)}{1+7cos(\theta)}d\theta = -\frac{1}{7}ln|1+7cos(\theta)| + C$

17) $\int \frac{ds}{tan^{-1}(s)+s^2tan^{-1}(s)} = \int \frac{1}{tan^{-1}(s)(1+s^2)}ds = ln|tan^{-1}(s)| + C$

18) $\int \frac{ln(x)}{4xln(2)}dx = \frac{1}{4ln(2)}\int \frac{ln(x)}{x}dx = \frac{\left(ln(x)\right)^2}{8ln(2)} + C$

19) $\int \frac{w^2+2w-1}{w+4}dw = \int \left(w - 2 + \frac{7}{w+4}\right)dw = \frac{w^2}{2} - 2w + 7ln|w+4| + C$

20) $\int \frac{ln(3)cos(x)}{-5-sin(x)}dx = -ln(3)ln|-5-sin(x)| + C$

8.5 Integrals of General Logarithmic Function

$$\frac{d}{du}\log_a(u) = \frac{1}{u \cdot \ln(a)}du \Rightarrow \int \frac{1}{u \cdot \ln(a)}du = \log_a(u) + C$$

where $a > 0$ and $a \neq 1$ (i.e., $\ln(a) \neq 0$)

Examples: Evaluate the following integrals:

1) $\int \frac{x}{x^2 ln5}dx = \frac{1}{2}\int \frac{2x}{x^2 ln5}dx = \frac{1}{2}\boldsymbol{log_5(x^2) + C}$

2) $\int \frac{cos(3t)}{sin(3t)ln4}dt = \frac{1}{3}\int \frac{3cos(3t)}{sin(3t)ln4}dt = \frac{1}{3}\boldsymbol{log_4\big(sin(3t)\big) + C}$

3) $\int \frac{1}{(\sqrt{1-x^2})sin^{-1}(x)ln3}dx = \boldsymbol{log_3\left(sin^{-1}(x)\right) + C}$

4) $\int \left(\sqrt{w} - \frac{1}{\sqrt{1-4w^2}cos^{-1}(2w)ln3}\right)dw = \frac{w^{3/2}}{3/2} - \int \frac{1}{\sqrt{1-(2w)^2}cos^{-1}(2w)ln3}dw =$

$\frac{2}{3}\boldsymbol{w^{3/2}} + \frac{1}{2}\boldsymbol{log_3\left(cos^{-1}(2w)\right) + C}$

Problems (8.5): Evaluate the following integrals:

1) $\int \frac{\cos(x)}{\sin(x)\ln3}dx = \mathbf{\log_3\big(\sin(x)\big) + C}$

2) $\int \frac{1}{e^{2x}\ln3} \cdot e^{2x}dx = \int \frac{1}{\ln3}dx = \mathbf{\frac{x}{ln3} + C}$

3) $\int \frac{1}{e^{\sin(x)}\ln7} \cdot e^{\sin(x)}\cos(x)dx = \int \frac{\cos(x)}{\ln7}dx = \mathbf{\frac{\sin(x)}{ln7} + C}$

4) $\int \frac{\frac{1}{1+x^2}}{\tan^{-1}(x)\ln11}dx = \mathbf{\log_{11}\left(\tan^{-1}(x)\right) + C}$

5) $\int \frac{x^3}{x^4\ln5}dx = \frac{1}{4}\int \frac{4x^3}{x^4\ln5}dx = \mathbf{\frac{1}{4}\log_5(x^4) + C}$

6) $\int \frac{1}{\cos^{-1}(w)\ln2}\frac{1}{\sqrt{1-w^2}}dw = \mathbf{-\log_2\left(\cos^{-1}(w)\right) + C}$

7) $\int \frac{1}{\cot^{-1}(x)\ln 4} \cdot \frac{1}{1+x^2}\,dx = \mathbf{-\log_4\left(\cot^{-1}(x)\right) + C}$

8) $\int \frac{1}{2t\ln 3}\,dt = \frac{1}{2}\int \frac{1}{t\ln 3}\,dt = \mathbf{\frac{1}{2}\log_3(t) + C}$

9) $\int \left(\cos(4z) - \frac{1}{\sqrt{1-z^2}\sin^{-1}(z)\ln 2}\right)dz = \mathbf{\frac{1}{4}\sin(4z) - \log_2\left(\sin^{-1}(z)\right) + C}$

10) $\int \frac{\sin(5w)}{\cos(5w)\ln 7}\,dw = -\frac{1}{5}\int \frac{-5\sin(5w)}{\cos(5w)\ln 7}\,dw = \mathbf{-\frac{1}{5}\log_7\left(\cos(5w)\right) + C}$