جامعة تكريت

كلية التربية البنات

قسم الرياضيات

# تفاضل وتكامل

## طرق التكامل

ا.م.د.هبه هاني عبدالله

hiba.h.a.83@tu.edu.iq

# CHAPTER 9: Techniques of Integration

## 9.1 Power of Trigonometric Functions

**A. To find the integral for:** $\int \sin^n(x)\cos^m(x)$

**case(1):** If one of the powers is **odd** (or both are odd), then we use the following form:

$$\mathbf{\sin^2(x) + \cos^2(x) = 1}$$

**case(2):** If both of the powers are **even**, then we use one of the following forms:

$$\mathbf{\sin^2(x) = \frac{1}{2}(1 - \cos 2x)} \quad \text{or} \quad \mathbf{\cos^2(x) = \frac{1}{2}(1 + \cos 2x)}$$

---

**Examples: Evaluate the following integrals:**

1) $\int \sin^2(x)\cos^3(x)dx$ [$n$:odd & $m$:even → case(1)]

$$= \int \sin^2(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^2(x)\cos(x)\left(1 - \sin^2(x)\right)dx$$

$$= \int \left(\sin^2(x)\cos(x) - \sin^4(x)\cos(x)\right)dx$$

$$= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

$$= \int \sin^2(x)\cos(x)dx - \int \sin^4(x)\cos(x)dx$$

$$= \mathbf{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

**2)** $\int \boldsymbol{\sin^3(x)\cos^3(x)dx}$ [$n$:odd & $m$:odd → case(1)]

$$= \int \sin^3(x)\cos(x)\cos^2(x)dx$$

$$= \int \sin^3(x)\cos(x)\left(1-\sin^2(x)\right)dx$$

$$= \int \sin^3(x)\cos(x)dx - \int \sin^5(x)\cos(x)dx$$

$$= \boldsymbol{\frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6} + C}$$

**3)** $\int \boldsymbol{\sin^2(x)\cos^2(x)dx}$ [$n$:even & $m$:even → case(2)]

$$= \int \frac{1}{2}\left(1-\cos(2x)\right)\frac{1}{2}\left(1+\cos(2x)\right)dx$$

$$= \int \frac{1}{4}\left(1-\cos^2(2x)\right)dx$$

$$= \int \frac{1}{4}dx - \int \frac{1}{4}\cos^2(2x)dx$$

$$= \int \frac{1}{4}dx - \int \frac{1}{4}\cdot\frac{1}{2}\left(1+\cos(4x)\right)dx$$

$$= \int \frac{1}{4}dx - \int \frac{1}{8}dx - \int \frac{1}{8}\cos(4x)dx$$

$$= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{32}\int \cos(4x)\cdot 4dx$$

$$= \boldsymbol{\frac{x}{4} - \frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

## B. To find the integral for: $\int \tan^n(x)\sec^m(x)$

**case(1):** If the powers $\underline{\sec(x)}$ is **even**, we divide the $\underline{\sec(x)}$ as:

$$\sec^m(x) = \sec^{m-2}(x)\sec^2(x) \ \& \qquad \text{use } \sec^2(x) = 1 + \tan^2(x)$$

---

**case(2):** If the powers $\tan(x)$ is **odd**, we divide the $\tan(x)$ as:

$$\tan^n(x) = \tan^{n-1}(x)\tan(x) \ \& \qquad \text{use } \tan^2(\mathrm{x}) = \sec^2(\mathrm{x}) - 1$$

---

**Examples: Evaluate the following integrals:**

1) $\int \sec^4(x)\tan^2(x)dx$

$$= \int \sec^2(x)\sec^2(x)\tan^2(x)dx$$

$$= \int \left(1 + \tan^2(x)\right)\sec^2(x)\tan^2(x)dx$$

$$= \int \tan^2(x)\sec^2(x)dx + \int \tan^4(x)\sec^2(x)dx$$

$$= \frac{\tan^3(\mathrm{x})}{3} + \frac{\tan^5(\mathrm{x})}{5} + C$$

2) $\int \tan^3(x)\sqrt{\sec(x)}dx$

$$= \int \tan^2(x)\tan(x)\sqrt{\sec(x)}dx$$

$$= \int (\sec^2(x) - 1)\tan(x)\sec^{\frac{1}{2}}(x)dx$$

$$= \int \sec^{\frac{3}{2}}(x)\big(\sec(x)\tan(x)\big)dx - \int \sec^{-\frac{1}{2}}(x)\big(\sec(x)\tan(x)\big)dx$$

$$= \int \sec^{\frac{3}{2}}(x)\sec(x)\tan(x)dx - \int \sec^{-\frac{1}{2}}(x)\sec(x)\tan(x)dx$$

$$= \frac{\sec^{\frac{5}{2}}(x)}{\frac{5}{2}} - \frac{\sec^{\frac{1}{2}}(x)}{\frac{1}{2}} + C$$

---

## C. To find the integral for: $\int \cot^n(x)\csc^m(x)$

case(1): If the power of $\underline{\csc(x)}$ is even, we divide the $\underline{\csc(x)}$ as:

$\csc^m(x) = \csc^{m-2}(x)\csc^2(x)$ **&** we use $\csc^2(x) = 1 + \cot^2(x)$

---

**case(2):** If the power of $\cot(x)$ is **odd**, we divide the $\cot(x)$ as:

$\cot^n(x) = \cot^{n-1}(x)\cot(x)$ **&** we use $\cot^2(x) = \csc^2(x) - 1$

**Examples:** **Evaluate the following integrals:**

1) $\int \csc^4(x)\cot^2(x)dx$

$$= \int \csc^2(x)\csc^2(x)\cot^2(x)dx$$

$$= \int \big(1 + \cot^2(x)\big)\csc^2(x)\cot^2(x)dx$$

$$= \int \cot^2(x)\csc^2(x)dx + \int \cot^4(x)\csc^2(x)dx$$

$$= -\frac{\cot^3(x)}{3} - \frac{\cot^5(x)}{5} + C$$

**2)** $\int \frac{\cot^3(x)}{\sqrt{\csc(x)}}dx$

$$= \int \cot^2(x)\cot(x)\csc^{-\frac{1}{2}}(x)dx$$

$$= \int (\csc^2(x) - 1)\cot(x)\csc^{-\frac{1}{2}}(x)dx$$

$$= \int \csc^{\frac{3}{2}}(x)\cot(x)dx - \int \cot(x)\csc^{-\frac{1}{2}}(x)dx$$

$$= \int \csc^{\frac{1}{2}}(x)\csc(x)\cot(x)dx - \int \csc^{-\frac{3}{2}}(x)\csc(x)\cot(x)dx$$

$$= \frac{\csc^{\frac{3}{2}}(x)}{\frac{3}{2}} - \frac{\csc^{-\frac{1}{2}}(x)}{\frac{-1}{2}} + C$$

---

**Problems (9.1): Evaluate the following integrals:**

**1)** $\int \cos^2(3\theta)d\theta = \int \frac{1}{2}\left(1+\cos(6\theta)\right)d\theta = \frac{1}{2}\theta + \frac{1}{12}\sin(6\theta) + C$

**2)** $\int \sin^4(2x)dx = \int \left[\frac{1}{2}\left(1-\cos(4x)\right)\right]^2 dx$

$$= \frac{3}{8}x - \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + C$$

**3)** $\int \cos^4(w)dw = \int \left[\frac{1}{2}\left(1+\cos(2w)\right)\right]^2 dw$

$$= \frac{3}{8}w + \frac{1}{4}\sin(2w) + \frac{1}{32}\sin(4w) + C$$

**4)** $\int \frac{3\tan^3(x)}{\sqrt{\sec(x)}}dx = \frac{6}{5}\sec^{\frac{5}{2}}(x) - 6\sec^{\frac{1}{2}}(x) + C$

**5)** $\int \cos^3(2x)dx = \int \left(1-\sin^2(2x)\right)\cos(2x)dx$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C$$

6) $\int \frac{\cos^3(t)}{\sin^2(t)}dt = \int \frac{1-\sin^2(t)}{\sin^2(t)}\cos(t)dt = -\frac{1}{\sin(t)} - \sin(t) + C$

7) $\int \frac{\sin^3(x)}{\cos^2(x)}dx = \int \frac{1-\cos^2(x)}{\cos^2(x)}\sin(x)dx = \frac{1}{\cos(x)} + \cos(x) + C$

8) $\int \frac{\cos(x)}{\left(1+\sin(x)\right)^2}dx = -\frac{1}{1+\sin(x)} + C$

9) $\int \sin^3(\theta)\cos^2(\theta)d\theta = \int \left(1-\cos^2(\theta)\right)\cos^2(\theta)\sin(\theta)d\theta$

$$= -\frac{\cos^3(\theta)}{3} + \frac{\cos^5(\theta)}{5} + C$$

10) $\int \sin^2(3x)\cos(3x)dx = \frac{\sin^3(3x)}{9} + C$

11) $\int \cos^{\frac{2}{3}}(3x)\sin^5(x)dx$

$$= \int \cos^{\frac{2}{3}}(3x)\left(1-\cos^2(3x)\right)^2\sin(3x)dx$$

$$= -\frac{1}{3}\left[\frac{3}{5}\cos^{\frac{5}{3}}(3x) - \frac{6}{11}\cos^{\frac{11}{3}}(3x) + \frac{3}{17}\cos^{\frac{17}{3}}(3x)\right] + C$$

12) $\int \sin^2(w)\cos^4(w)dw = \frac{w}{16} - \frac{\sin(4w)}{64} - \frac{\sin^3(2w)}{48} + C$

13) $\int \frac{\cos(2t)}{\sin^4(t)}dt = \int \frac{1-2\sin^2(t)}{\sin^4(t)}dt = -\frac{1}{3\sin^3(t)} + \frac{2}{\sin(t)} + C$

14) $\int \cot^3(\theta)d\theta = \int \cot(\theta)(\csc^2(\theta)-1)d\theta$

$$= -\frac{1}{2}\cot^2(\theta) - \ln|\sin(\theta)| + C$$

15) $\int \tan^2(4t)dt = \int (\sec^2(4t)-1)dt = \frac{1}{4}\tan(4t) - t + C$

16) $\int \boldsymbol{\csc^4(x)dx} = \int \csc^2(x)\left(1+\cot^2(x)\right)dx$

$$= -\mathbf{cot(x)} - \frac{\mathbf{1}}{\mathbf{3}}\mathbf{cot^3(x)} + \mathbf{C}$$

17) $\int \boldsymbol{\csc^2(2z)\cot(2z)dz} = -\frac{1}{4}\mathbf{cot^2(2z)} + \mathbf{C}$

18) $\int \boldsymbol{\sec^4(3x)\tan(3x)dx} = \frac{1}{12}\mathbf{sec^4(3x)} + \mathbf{C}$

19) $\int \frac{1}{\cos^2(w)}\boldsymbol{dw} = \int \sec^2(w)dw = \mathbf{tan(w)} + \mathbf{C}$

20) $\int \boldsymbol{\tan^3(x)\sec(x)dx} = \int (\sec^2(x)-1)\sec(x)\tan(x)dx$

$$= \frac{\mathbf{1}}{\mathbf{3}}\mathbf{sec^3(x)} - \mathbf{sec(x)} + \mathbf{C}$$

---

## 9.2 Integration by Parts

$\because d(u.v) = ud(v) + vd(u)$

$\Longrightarrow ud(v) = d(u.v) - vd(u)$

$\xrightarrow{\int} \int ud(v) = \int d(u.v) - \int vd(u) \Longrightarrow \int \mathbf{ud(v)} = \mathbf{u.v} - \int \mathbf{vd(u)}$

---

**Examples:** **Evaluate the following integrals:**

1) $\int \boldsymbol{\ln(x)dx} = ?$

let $u = \ln(x) \Longrightarrow du = \frac{dx}{x}$

$dv = dx \Longrightarrow v = \int dx = x$

$\because \int ud(v) = u.v - \int vd(u)$

$$\Longrightarrow \int \ln(x)dx = \ln(x).x - \int x.\frac{dx}{x}$$

$= \ln(x).x - \int dx = \mathbf{\ln(x).x - x + C}$

**2)** $\int \boldsymbol{xe^x dx} = ?$

let $u = x \Longrightarrow du = dx$

$dv = e^x dx \Longrightarrow v = \int e^x dx = e^x$

$\because \int ud(v) = u.v - \int vd(u)$

$\Longrightarrow \int xe^x dx = x.e^x - \int e^x dx$

$= \mathbf{xe^x - e^x + C}$

**3)** $\int \boldsymbol{x^2 e^x dx} = ?$

let $u = x^2 \Longrightarrow du = 2xdx\ dv = e^x dx \Longrightarrow v = \int e^x dx = e^x$

$\because \int udv = u \cdot v - \int vdu \Longrightarrow \int x^2 e^x dx = x^2 \cdot e^x - \int e^x \cdot 2xdx$

$= x^2 \cdot e^x - 2\int e^x \cdot xdx = x^2 \cdot e^x - 2(xe^x - e^x + C)$

$= \boldsymbol{x^2 e^x - 2xe^x + 2e^x + C}$

**4)** $\int \boldsymbol{e^x \sin(2x) dx} = ?$

let $u = e^x \Longrightarrow du = e^x dx$

$dv = \sin(2x)dx \Longrightarrow v = \frac{1}{2}\int \sin(2x) \cdot 2dx = -\frac{1}{2}\cos(2x)$

$\because \int udv = u \cdot v - \int vdu$

$\Longrightarrow \int e^x \sin(2x)dx = e^x \cdot -\frac{1}{2}\cos(2x) - \int -\frac{1}{2}\cos(2x) \cdot e^x dx \Longrightarrow$

$\int e^x \sin(2x)dx = -\frac{1}{2}e^x \cos(2x) + \frac{1}{2}\int \cos(2x) \cdot e^x dx$

let $u = e^x \Longrightarrow du = e^x dx$

$dv = \cos(2x)dx \Longrightarrow v = \frac{1}{2}\sin(2x)$

$\because \int \cos(2x) \cdot e^x dx = e^x \cdot \frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)e^x dx$

$$\Longrightarrow \int e^x \sin(2x)dx = e^x \cdot \frac{-1}{2}\cos(2x) + \frac{1}{4}e^x \cdot \sin(2x) - \frac{1}{4}\int e^x \sin(2x)dx$$

$$\Longrightarrow \frac{5}{4}\int e^x \sin(2x)dx = \frac{-1}{2}e^x \cdot \cos(2x) + \frac{1}{4}e^x \cdot \sin(2x)$$

$$\overset{*\frac{4}{5}}{\Longrightarrow} \int \boldsymbol{e^x \sin(2x)dx} = \frac{\mathbf{-2}}{\mathbf{5}}\boldsymbol{e^x} \cdot \mathbf{cos}(\mathbf{2}\boldsymbol{x}) + \frac{\mathbf{1}}{\mathbf{5}}\boldsymbol{e^x} \cdot \mathbf{sin}(\mathbf{2}\boldsymbol{x}) + \boldsymbol{C}$$

**5)** $\int \frac{\boldsymbol{x^3}}{(\boldsymbol{x^2}+\mathbf{1})^{\frac{3}{2}}}\boldsymbol{dx} = ?$

$$\int \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = \int x^2 \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$$

let $u = x^2 \Longrightarrow dv = 2xdx$

$$dv = \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$$

$$\Longrightarrow v = \int \frac{x}{(x^2+1)^{\frac{3}{2}}}dx = \int x \cdot (x^2+1)^{-\frac{3}{2}}dx$$

$$= \frac{1}{2}\int 2x \cdot (x^2+1)^{-\frac{3}{2}}dx = \frac{1}{2}\frac{(x^2+1)^{-\frac{1}{2}}}{-\frac{1}{2}} = \frac{-1}{(x^2+1)^{\frac{1}{2}}}$$

$$\int ud(v) = u \cdot v - \int vdu$$

$$\Longrightarrow \int \frac{x^3}{(x^2+1)^{\frac{3}{2}}}dx = x^2 \cdot \frac{-1}{(x^2+1)^{\frac{1}{2}}} - \int \frac{-1}{(x^2+1)^{\frac{1}{2}}} \cdot 2xdx$$

$$= \frac{-x^2}{(x^2+1)^{\frac{1}{2}}} + \int (x^2+1)^{-\frac{1}{2}} \cdot 2xdx$$

$$= \frac{-\boldsymbol{x^2}}{(\boldsymbol{x^2}+\mathbf{1})^{\frac{1}{2}}} + \frac{(\boldsymbol{x^2}+\mathbf{1})^{\frac{1}{2}}}{\frac{1}{2}} + \boldsymbol{C}$$

**6)** $\int \mathbf{sec^3}(\boldsymbol{x})\boldsymbol{dx} = ?$

$$\int \sec^3(x)dx = \int \sec(x)_u \sec^2(x)_{dv} dx$$

$$\text{let } u = \sec(x) \Longrightarrow du = \sec(x)\tan(x)dx$$

$$dv = \sec^2(x)dx \Longrightarrow v = \int \sec^2(x)dx = \tan(x)$$

$$\because \int udv = u \cdot v - \int vdu$$

$$\Longrightarrow \int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)dx$$

$$= \sec(x)\tan(x) - \int \tan^2(x)\sec(x)dx$$

$$= \sec(x)\tan(x) - \int [\sec^2(x) - 1]\sec(x)dx$$

$$\Longrightarrow \int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx$$

$$\Longrightarrow 2\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$$

$$\Longrightarrow \int \boldsymbol{sec^3}(\boldsymbol{x})\boldsymbol{dx} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{sec}\,(\boldsymbol{x})\boldsymbol{tan}\,(\boldsymbol{x}) + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{ln}\,|\boldsymbol{sec}\,(\boldsymbol{x}) + \boldsymbol{tan}\,(\boldsymbol{x})| + \boldsymbol{K}$$

**Problems (9.2): Evaluate the following integrals:**

1) $\int e^{\ln\sqrt{x}}dx$

$$= \int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$$

2) $\int \ln\sqrt{x-1}dx$

$$= \frac{1}{2}\int \ln(x-1)dx$$

$$u = \ln(x-1) \Longrightarrow du = \frac{1}{x-1}dx, \text{quad} dv = dx \Longrightarrow v = x$$

$$= \frac{1}{2}[x\ln(x-1) - \int \frac{x}{x-1}dx] = \frac{1}{2}[x\ln(x-1) - (x + \ln|x-1|)] + C$$

3) $\int e^{x}\cos(2x)dx$

$$= \frac{1}{5}e^{x}(\cos(2x) + 2\sin(2x)) + C$$

4) $\int x\sec^2(x)dx$

$$u = x \Longrightarrow du = dx, \text{quad} dv = \sec^2(x)dx \Longrightarrow v = \tan(x)$$

$$= x\tan(x) - \int \tan(x)dx = x\tan(x) + \ln|\cos(x)| + C$$

5) $\int \ln(x + \sqrt{1+x^2})dx$

$$u = \ln(x + \sqrt{1+x^2}) \Longrightarrow du = \frac{1}{\sqrt{1+x^2}}dx, \text{quad} dv = dx \Longrightarrow v = x$$

$$= x\ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}dx = x\ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

**6)** $\int \boldsymbol{x\tan^2(x)dx}$

$$= \int x(\sec^2(x) - 1)dx = \int x\sec^2(x)dx - \int x dx$$

$$= x\tan(x) + \ln|\cos(x)| - \frac{x^2}{2} + C$$

**7)** $\int \frac{\tan^{-1}(x)}{x^2} \boldsymbol{dx}$

$$u = \tan^{-1}(x) \Longrightarrow du = \frac{1}{1+x^2}dx, \qquad dv = x^{-2}dx \Longrightarrow v = -\frac{1}{x}$$

$$= -\frac{\tan^{-1}(x)}{x} + \int \frac{1}{x(1+x^2)}dx = -\frac{\tan^{-1}(x)}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$$

**8)** $\int \boldsymbol{x\cos^2(x)dx}$

$$= \int x(\frac{1+\cos(2x)}{2})dx = \frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + C$$

**9)** $\int \boldsymbol{x^2\sin(x)dx}$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

**10)** $\int \boldsymbol{z\sin^2(z)dz}$

$$= \frac{z^2}{4} - \frac{z\sin(2z)}{4} - \frac{\cos(2z)}{8} + C$$

**11)** $\int \boldsymbol{x\ln\sqrt{x+2}dx}$

$$= \frac{1}{2}\int x\ln(x+2)dx$$

$$= \frac{1}{2}[\frac{x^2-4}{2}\ln(x+2) - \frac{x^2}{4} + x] + C$$

**12)** $\int (\boldsymbol{x}+\mathbf{1})^{\mathbf{2}}\boldsymbol{e^{x}dx}$

$= e^{x}(x^{2}+2x+1-2x-2+2) = e^{x}(x^{2}+1)+C$

**13)** $\int \boldsymbol{e^{2t}}\mathbf{cos}(\boldsymbol{e^{t}})\boldsymbol{dt}$

let $w = e^{t} \Longrightarrow dw = e^{t}dt$

$= \int w\cos(w)dw = w\sin(w)+\cos(w)+C = e^{t}\sin(e^{t})+\cos(e^{t})+C$

**14)** $\int \boldsymbol{x}\mathbf{ln}(\boldsymbol{x^{3}}+\boldsymbol{x})\boldsymbol{dx}$

$$= \frac{x^{2}}{2}\ln(x^{3}+x) - \int \frac{3x^{2}+1}{2(x^{2}+1)}dx = \frac{x^{2}}{2}\ln(x^{3}+x) - \frac{3}{2}x + \tan^{-1}(x) + C$$

**15)** $\int \boldsymbol{x^{3}e^{x^{2}}dx}$

let $w = x^{2} \Longrightarrow dw = 2xdx$

$$= \frac{1}{2}\int we^{w}dw = \frac{1}{2}(we^{w}-e^{w})+C = \frac{1}{2}e^{x^{2}}(x^{2}-1)+C$$

**16)** $\int \boldsymbol{x^{2}}\mathbf{sin}(\mathbf{1}-\boldsymbol{x})\boldsymbol{dx}$

$= x^{2}\cos(1-x)+2x\sin(1-x)-2\cos(1-x)+C$

**17)** $\int \boldsymbol{z}\mathbf{sin^{2}}(\boldsymbol{z})\boldsymbol{dz}$

$$= \frac{z^{2}}{4} - \frac{z\sin(2z)}{4} - \frac{\cos(2z)}{8} + C$$

**18)** $\int \boldsymbol{x}\mathbf{ln}(\sqrt[3]{\mathbf{3}\boldsymbol{x}+\mathbf{1}})\boldsymbol{dx}$

$$= \frac{1}{3}\int x\ln(3x+1)dx = \frac{1}{3}[\frac{x^{2}}{2}\ln(3x+1) - \int \frac{3x^{2}}{2(3x+1)}dx]$$

$$= \frac{x^2}{6}\ln(3x+1) - \frac{1}{6}(\frac{x^2}{2} - \frac{x}{3} + \frac{1}{9}\ln|3x+1|) + C$$

**19)** $\int \boldsymbol{\cos(\ln x)dx}$

$$= \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$$

**20)** $\int \boldsymbol{e^{2x}\sin(3x)dx}$

$$= \frac{e^{2x}}{13}(2\sin(3x) - 3\cos(3x)) + C$$

**21)** $\int \boldsymbol{x\sin(x)dx}$

$$= -x\cos(x) + \sin(x) + C$$

**22)** $\int \boldsymbol{\theta\cos(3\theta)d\theta}$

$$= \frac{\theta}{3}\sin(3\theta) + \frac{1}{9}\cos(3\theta) + C$$

**23)** $\int \boldsymbol{x\sqrt{4x+2}dx}$

$$= \frac{1}{10}(4x+2)^{\frac{5}{2}} - \frac{1}{12}(4x+2)^{\frac{3}{2}} + C$$

**24)** $\int \boldsymbol{x tan^{-1} x dx}$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + C$$